

QUANTUM HAMILTONIAN REDUCTION OF W-ALGEBRAS AND CATEGORY \mathcal{O}

STEPHEN MORGAN

ABSTRACT. We define a quantum version of Hamiltonian reduction by stages, producing a construction in type A for a quantum Hamiltonian reduction from the W-algebra $U(\mathfrak{g}, e_1)$ to an algebra conjecturally isomorphic to $U(\mathfrak{g}, e_2)$, whenever $e_2 \geq e_1$ in the dominance ordering. This isomorphism is shown to hold whenever e_1 is subregular, and in \mathfrak{sl}_n for all $n \leq 4$.

We next define embeddings of various categories \mathcal{O} for the W-algebras associated to e_1 and e_2 , amongst them the embeddings $\mathcal{O}(e_2, \mathfrak{p}) \hookrightarrow \mathcal{O}(e_1, \mathfrak{p})$, where \mathfrak{p} is a parabolic subalgebra containing both e_1 and e_2 in its Levi subalgebra.

1. INTRODUCTION

Let \mathfrak{g} be a semi-simple complex Lie algebra with universal enveloping algebra $U(\mathfrak{g})$, and let G be the simply-connected algebraic group satisfying $\mathrm{Lie}(G) = \mathfrak{g}$. To any nilpotent element $e \in \mathfrak{g}$, one can associate a certain non-commutative algebra $U(\mathfrak{g}, e)$ known as the *W-algebra* associated to the nilpotent e . There are several definitions of the W-algebra depending on different choices and parameters, but it is known that they are all equivalent up to isomorphism, and depend only on the nilpotent orbit of e under the adjoint action of G .

The definition of W-algebras of primary use in this paper is as a *quantum Hamiltonian reduction* of the universal enveloping algebra $U(\mathfrak{g})$ with respect to a choice of nilpotent subalgebra \mathfrak{m} and character χ thereof, both derived from the nilpotent e . In short, given a Lie algebra \mathfrak{m} coming from a good grading of \mathfrak{g} and the character $\chi \in \mathfrak{m}^*$ associated to e under the identification $\mathfrak{g} \simeq \mathfrak{g}^*$ given by the Killing form. Considering the shift $\mathfrak{m}_\chi := \{y - \chi(y) : y \in \mathfrak{m}\}$, the W-algebra can be defined as the algebra of invariants in the quotient $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi$ under the adjoint action of \mathfrak{m} ; i.e. $U(\mathfrak{g}, e) := (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi)^{\mathfrak{m}}$. Since all objects involved are filtered by the Kazhdan filtration, the W-algebra is itself a filtered algebra. Taking the associated graded algebra yields the ring of functions on the Slodowy slice $\mathcal{S}_\chi \subseteq \mathfrak{g}^*$, and quantum Hamiltonian reduction reduces to ordinary Hamiltonian reduction of Slodowy slices [CG].

With this framework in mind, one can ask whether this quantum Hamiltonian reduction can be decomposed into steps, analogous to the classical construction of *Hamiltonian reduction by stages*. In particular, given a pair of W-algebras defined by quantum Hamiltonian reduction, when can an intermediate reduction between the two be found which commutes with the original reductions up to isomorphism, as in fig. 1.

In section 3, we give a partial answer to this question in type A. We first present a construction using the combinatorics of *pyramids*, which for any pair of nilpotent elements $e_1, e_2 \in \mathfrak{sl}_n$, where e_2 covers e_1 in the dominance ordering, produces an intermediate reduction from the W-algebra $U(\mathfrak{g}, e_1)$ in type A to a certain algebra

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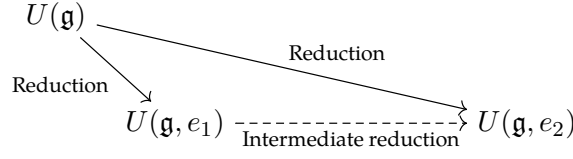


FIGURE 1. Reduction of W-algebras by stages.

associated to e_2 . We conjecture that this algebra is isomorphic to the W-algebra $U(\mathfrak{g}, e_2)$ (conjecture 3.13), and present a proof in known cases.

In section 4, we turn our attention to the representation theory of W-algebras. This is a subject which has been widely studied, and a number of connections to the representation theory of the Lie algebra \mathfrak{g} itself have been found. The construction of quantum Hamiltonian reduction by stages produces a $(U(\mathfrak{g}, e_1), U(\mathfrak{g}, e_2))$ -bimodule for any pair of nilpotents e_1 and e_2 as above. This in turn provides a pair of adjoint functors $U(\mathfrak{g}, e_1)\text{-mod} \rightleftarrows U(\mathfrak{g}, e_2)\text{-mod}$ for any such pair. A modification of an argument of Loseu [Los3] can be used to produce embeddings of the corresponding categories $\mathcal{O}(e_2, \mathfrak{p}) \hookrightarrow \mathcal{O}(e_1, \mathfrak{p})$, whenever $e_1 \leq e_2$.

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2. W-ALGEBRAS AND QUANTUM HAMILTONIAN REDUCTION

We first recall the definition of W-algebras via quantum Hamiltonian reduction. Let \mathfrak{g} be a semisimple Lie algebra over \mathbb{C} , and let $e \in \mathfrak{g}$ be a chosen nilpotent element. By the Jacobson–Morozov theorem any non-zero e be completed to an \mathfrak{sl}_2 -triple (e, h, f) . The semisimple element h determines a \mathbb{Z} -grading of the Lie algebra \mathfrak{g} by declaring $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, where $\mathfrak{g}_j = \{x \in \mathfrak{g} : [h, x] = jx\}$. This grading satisfies the following list of useful properties, where $\mathfrak{z}_{\mathfrak{g}}(e)$ is the centraliser of e in \mathfrak{g} , and $\langle \cdot, \cdot \rangle$ is the Killing form:

- (GG1) $e \in \mathfrak{g}_2$,
- (GG2) $\text{ad } e: \mathfrak{g}_j \rightarrow \mathfrak{g}_{j+2}$ is injective for $j \leq -1$,
- (GG3) $\text{ad } e: \mathfrak{g}_j \rightarrow \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$,
- (GG4) $\mathfrak{z}_{\mathfrak{g}}(e) \subseteq \bigoplus_{j \geq 0} \mathfrak{g}_j$,
- (GG5) $\langle \mathfrak{g}_i, \mathfrak{g}_j \rangle = 0$ unless $i + j = 0$,
- (GG6) $\dim \mathfrak{z}_{\mathfrak{g}}(e) = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$.

It is a well-known result that any grading which satisfies properties GG1 to GG3 will necessarily satisfy all of them (and even more strongly that properties GG2 and GG3 are equivalent for any \mathbb{Z} -grading). This motivates the following definition, which provides a generalisation of the gradings coming from \mathfrak{sl}_2 -triples.

Definition 2.1. A \mathbb{Z} -grading of \mathfrak{g} is called a *good grading* for the nilpotent e if it satisfies properties GG1 to GG3. A good grading which comes from an \mathfrak{sl}_2 -triple containing e is called a *Dynkin grading*. A good grading which vanishes in odd degree is called an *even grading*.

Note that although all Dynkin gradings are good, there exist good gradings which are not Dynkin: these non-Dynkin good gradings will be important for our work. From this point on, fix a good grading of the Lie algebra \mathfrak{g} .

The space \mathfrak{g}_{-1} has a natural symplectic form ω given by $\omega(x, y) := \langle e, [x, y] \rangle$. Choosing a Lagrangian subspace $\mathfrak{l} \subseteq \mathfrak{g}_{-1}$ with respect to this form, one can define a Premet subalgebra $\mathfrak{m} := \mathfrak{l} \oplus \bigoplus_{j \leq -2} \mathfrak{g}_j$. Premet subalgebras enjoy a number of properties we record for future reference. Let \mathcal{O}_e be the adjoint orbit through e .

- (χ 1) \mathfrak{m} is an ad-nilpotent subalgebra of \mathfrak{g} ,
- (χ 2) $\dim \mathfrak{m} = \frac{1}{2} \dim \mathcal{O}_e$,
- (χ 3) $\mathfrak{m} \cap \mathfrak{z}_{\mathfrak{g}}(e) = 0$,
- (χ 4) $\chi := \langle e, \cdot \rangle$ restricts to a character of \mathfrak{m} ,

Given a Lie algebra \mathfrak{m} with character χ , one can define the shifted Lie algebra $\mathfrak{m}_{\chi} := \{y - \chi(y) : y \in \mathfrak{m}\}$. With this in hand, we can define the W-algebra.

Definition 2.2. Let $e \in \mathfrak{g}$ be a nilpotent element with a chosen good grading and Lagrangian subspace $\mathfrak{l} \subseteq \mathfrak{g}_{-1}$, and let \mathfrak{m} be the associated Premet subalgebra. The (finite) W-algebra $U(\mathfrak{g}, e)$ is the set of invariants in the quotient $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_{\chi}$, under the adjoint action of \mathfrak{m} :

$$U(\mathfrak{g}, e) := (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_{\chi})^{\mathfrak{m}} = \{\bar{u} \in U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_{\chi} : [a, u] \in U(\mathfrak{g})\mathfrak{m}_{\chi} \ \forall a \in \mathfrak{m}\}.$$

2.1. Slodowy slices. For any nilpotent element $e \in \mathfrak{g}$, one can construct an \mathfrak{sl}_2 -triple (e, h, f) by the Jacobson–Morozov theorem. In fact, any such pair of triples (e, h, f) and (e', h', f') for which $\mathcal{O}_e = \mathcal{O}_{e'}$ are conjugate by the adjoint action, and so in particular one can speak unambiguously of “the \mathfrak{sl}_2 -triple associated to e ”. If one additionally has a good grading Γ for the nilpotent e , the \mathfrak{sl}_2 -triple can be chosen to be Γ -graded, so that e, h and f lie in graded degrees 2, 0 and -2, respectively.

Associated to an \mathfrak{sl}_2 -triple (e, h, f) is a certain subvariety $\mathcal{S}_e \subseteq \mathfrak{g}$ known as the *Slodowy slice*. It is an affine space which forms a transverse slice to the nilpotent orbit \mathcal{O}_e , and is defined as a translate of the centraliser of f . We shall usually deal with the Slodowy slice in the dual Lie algebra \mathfrak{g}^* , transported via the Killing isomorphism $\kappa: \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$.

$$\mathcal{S}_e := e + \mathfrak{z}_{\mathfrak{g}}(f) \subseteq \mathfrak{g} \qquad \mathcal{S}_{\chi} := \chi + (\mathfrak{g}/[\mathfrak{g}, f])^* = \kappa(\mathcal{S}_e) \subseteq \mathfrak{g}^*$$

The Slodowy slice \mathcal{S}_{χ} , and hence \mathcal{S}_e , inherits a natural Poisson structure from the variety \mathfrak{g}^* equipped with the Lie–Poisson bracket.

Slodowy slices are of independent interest, but for our purposes we concentrate on their relation to W-algebras. Gan and Ginzburg proved in [GG] that $U(\mathfrak{g}, e)$ has the structure of a filtered algebra, and that the corresponding associated graded algebra $\text{gr } U(\mathfrak{g}, e)$ is the ring of functions on the Slodowy slice $\mathbb{C}[\mathcal{S}_{\chi}]$. Further, taking M to be the algebraic group with Lie algebra \mathfrak{m} , one can consider the moment map of the co-adjoint action of M on \mathfrak{g}^* : this is just restriction map $\mu: \mathfrak{g}^* \rightarrow \mathfrak{m}^*$. The Slodowy slice \mathcal{S}_{χ} can be expressed as a Hamiltonian reduction of \mathfrak{g}^* :

$$\mathcal{S}_{\chi} \simeq \mathfrak{g}^* //_{\chi} M := \mu^{-1}(\chi)/M.$$

Expressed in terms of rings of functions, and taking I_{χ} to be the ideal of functions which vanish on $\mu^{-1}(\chi)$, this takes the form

$$(1) \qquad \mathbb{C}[\mathcal{S}_{\chi}] \simeq (\mathbb{C}[\mathfrak{g}^*]/I_{\chi})^M.$$

The Poisson structure which \mathcal{S}_{χ} inherits as a Hamiltonian reduction of \mathfrak{g}^* agrees with the Poisson structure it inherits as a subvariety of \mathfrak{g}^* (cf. [GG, Section 3.4]). The similarity between definition 2.2 and eq. (1) has led to definition 2.2 to be referred to as a *quantum Hamiltonian reduction*, by analogy. This can be formalised in the language of deformation quantisations.

2.2. Deformation quantisations and quantum Hamiltonian reduction.

Definition 2.3. Let A be a Poisson algebra with Poisson bracket $\{\cdot, \cdot\}$. A *deformation quantisation* of A is an associative unital product $\star: A \otimes A \rightarrow A[[\hbar]]$ such that, when extended $\mathbb{C}[[\hbar]]$ -bilinearly, satisfies the following conditions:

- (1) \star is an associative binary product on $A[[\hbar]]$, continuous in the \hbar -adic topology;
- (2) $f \star g = fg + O(\hbar)$ for all $f, g \in A$;
- (3) $f \star g - g \star f = \{f, g\}\hbar + O(\hbar^2)$ for all $f, g \in A$.

Writing $f \star g = \sum_{k \geq 0} D_k(f, g)\hbar^k$, we shall further require that \star be a *differential deformation quantisation*, that is one satisfying the additional condition:

- (4) for each k , $D_k(\cdot, \cdot)$ is a bidifferential operator of order at most k in each variable.

The vector space $A[[\hbar]]$ equipped with the multiplication \star shall be denoted \mathcal{A}_\hbar , and is often referred to as a deformation quantisation itself. If X is a Poisson variety and A is its ring of functions, we often say that \mathcal{A}_\hbar is a deformation quantisation of X .

The product \star can also be used to introduce a new associative product on A through the projection $A[[\hbar]] \rightarrow A$, given by sending \hbar to 1. More concretely, define the product $\circ: A \otimes A \rightarrow A$ by $f \circ g := \sum_{k \geq 0} D_k(f, g)$. Let the vector space A equipped with this new algebra structure be denoted \mathcal{A} . By abuse of terminology, the algebra \mathcal{A} is often referred to as a deformation quantisation of A .

By results of Gan and Ginzburg [GG] and Loseu [Los1], the Rees algebra of the W-algebra considered with the Kazhdan filtration, denoted $U_\hbar(\mathfrak{g}, e)$, is a deformation quantisation of the ring of functions of the Slodowy slice $A = \mathbb{C}[\mathcal{S}_\chi]$. The W-algebra $U(\mathfrak{g}, e)$ itself is then just \mathcal{A} .

As a special case, we consider the $\mathbb{C}[[\hbar]]$ -extended universal enveloping algebra $U_\hbar(\mathfrak{g})$, which is a deformation quantisation of $\mathbb{C}[\mathfrak{g}^*]$. Consider the vector space $\mathfrak{g}_\hbar := \mathfrak{g} \otimes \mathbb{C}[[\hbar]]$ equipped with the Lie bracket $[\cdot, \cdot]_\hbar$, defined as $[x, y]_\hbar := [x, y]\hbar$ for $x, y \in \mathfrak{g}$ and extended $\mathbb{C}[[\hbar]]$ -bilinearly. The algebra $U_\hbar(\mathfrak{g})$ is then the universal enveloping algebra of \mathfrak{g}_\hbar , and can be concretely presented as the tensor algebra $T(\mathfrak{g}_\hbar)$ modulo the relation $xy - yx = [x, y]_\hbar$. This algebra $U_\hbar(\mathfrak{g})$ is just the Rees algebra of $U(\mathfrak{g})$ considered with the PBW filtration.

Assume now that G is an algebraic group which acts on \mathcal{A}_\hbar by $\mathbb{C}[[\hbar]]$ -algebra automorphisms, and preserves the grading. This induces an action of \mathfrak{g} on \mathcal{A}_\hbar by derivations, and we denote the derivation corresponding to $\xi \in \mathfrak{g}$ by $\xi_{\mathcal{A}}$. Let there furthermore exist a *quantum comoment map* for the action of G on \mathcal{A}_\hbar , i.e. a linear map $\Phi: \mathfrak{g} \rightarrow \mathcal{A}_\hbar$, which is G -equivariant and satisfies $\frac{1}{\hbar}[\Phi(\xi), \cdot] = \xi_{\mathcal{A}}$. It shall be useful to extend this $\mathbb{C}[[\hbar]]$ -linearly to a map $\Phi: U_\hbar(\mathfrak{g}) \rightarrow \mathcal{A}_\hbar$.

Definition 2.4. Let \mathcal{A}_\hbar be a deformation quantisation on which G acts with quantum comoment map Φ . Let $\gamma \in \mathfrak{g}^*$ be fixed under the co-adjoint action of G , and define \mathcal{I}_γ as the two-sided ideal in $U_\hbar(\mathfrak{g})$ generated by $\mathfrak{g}_{\hbar, \gamma} := \{x - \gamma(x)\hbar : x \in \mathfrak{g}\}$. The *quantum Hamiltonian reduction* of \mathcal{A}_\hbar at γ under the action of G is

$$\mathcal{A}_\hbar //_\gamma G := (\mathcal{A}_\hbar / \mathcal{A}_\hbar \Phi(\mathcal{I}_\gamma))^G.$$

This has a natural algebra structure with multiplication given by

$$(a + \mathcal{A}_\hbar \Phi(\mathcal{I}_\gamma))(b + \mathcal{A}_\hbar \Phi(\mathcal{I}_\gamma)) = ab + \mathcal{A}_\hbar \Phi(\mathcal{I}_\gamma).$$

Remark 2.5. Let \mathcal{A}_\hbar be a deformation quantisation of the Poisson variety X , and let G act on \mathcal{A}_\hbar with quantum comoment map Φ . Assume further that the action of G on \mathcal{A}_\hbar is induced by an action of G on X . Then the quantum comoment map Φ induces a classical moment map $\mu: X \rightarrow \mathfrak{g}^*$, and for any $\gamma \in \mathfrak{g}^*$ fixed under the

co-adjoint action of G , the quantum Hamiltonian reduction $\mathcal{A}_\hbar //_\gamma G$ is a deformation quantisation of the classical Hamiltonian reduction $X //_\gamma G$.

We shall now give a justification for calling definition 2.2 the definition of the W-algebra by quantum Hamiltonian reduction. Choosing a Premet subalgebra \mathfrak{m} for e naturally produces an algebraic group $M \subseteq G$ by exponentiation, since \mathfrak{m} is an ad-nilpotent subalgebra (χ1). This acts on \mathfrak{g}^* by the restriction of the co-adjoint action, and on $U_\hbar(\mathfrak{g})$ by extending the adjoint action. Furthermore, this action has a quantum comoment map $\Phi: \mathfrak{m} \rightarrow U_\hbar(\mathfrak{g})$ which comes from the natural inclusion of \mathfrak{m} into \mathfrak{g} , and extends to the natural inclusion of $U_\hbar(\mathfrak{m})$ into $U_\hbar(\mathfrak{g})$.

Since $\chi \in \mathfrak{m}^*$ is a character of \mathfrak{m} (χ4) it is fixed under the co-adjoint action of M , and so we can consider the quantum Hamiltonian reduction $U_\hbar(\mathfrak{g}) //_\chi M$. Since M is a unipotent algebraic group, invariants under adjoint action of M are completely equivalent to invariants under the adjoint action of \mathfrak{m} . As a result,

$$U_\hbar(\mathfrak{g}) //_\chi M := (U_\hbar(\mathfrak{g}) / U_\hbar(\mathfrak{g})\Phi(\mathcal{I}_\chi))^M = (U_\hbar(\mathfrak{g}) / U_\hbar(\mathfrak{g})\mathcal{I}_\chi)^\mathfrak{m},$$

and passing through the projection $\hbar \mapsto 1$ results in the definition of the W-algebra given in definition 2.2. We can therefore without ambiguity denote the above by $U(\mathfrak{g}) //_\chi \mathfrak{m}$.

2.3. Quantum Hamiltonian reduction by stages. Consider the Slodowy slice associated to the zero nilpotent, $\mathcal{S}_0 = \mathfrak{g}^*$. Equation (1) can then be restated in the following way: the Slodowy slice \mathcal{S}_χ can be expressed as a Hamiltonian reduction of \mathcal{S}_0 . In order to answer for which other pairs of nilpotent elements e_1 and e_2 this can be done, we need to introduce the machinery of *Hamiltonian reduction by stages*.

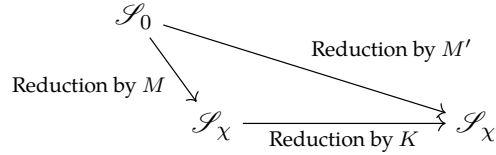


FIGURE 2. Hamiltonian reduction of Slodowy slices by stages

Reduction by stages is a technique for decomposing a Hamiltonian reduction into a sequence of smaller reductions. The general theory is quite highly developed (cf. [MMO⁺]), but we shall be interested in the specific case of reduction by a semidirect product.

Let $G \simeq H \rtimes K$ be an algebraic group which is a semidirect product of the closed subgroups H and K , with H normal in G . Let X be a Poisson variety with a Hamiltonian action of G with equivariant moment map $\mu: X \rightarrow \mathfrak{g}^*$. Let $\gamma \in \mathfrak{g}^*$ be a regular value of μ , which is identified with (η, κ) under the decomposition $\mathfrak{g}^* = \mathfrak{h}^* \times \mathfrak{k}^*$. Under certain mild conditions on the subgroup K and the values η and κ , there exists an isomorphism of Poisson varieties

$$X //_\gamma G \simeq (X //_\eta H) //_\kappa K,$$

where all the induced actions are well-defined and Hamiltonian.

With this in mind, we seek to define an analogous notion of *quantum Hamiltonian reduction by stages*.

Theorem 2.6. *Let \mathcal{A}_\hbar be a deformation quantisation, and let $G \simeq H \rtimes K$ be an algebraic group which acts on it with quantum comoment map $\Phi: \mathfrak{g} \rightarrow \mathcal{A}_\hbar$. Let $\gamma \in \mathfrak{g}^*$ be an invariant under the co-adjoint action of G , which decomposes as $\gamma = (\eta, \kappa)$ under the identification*

$\mathfrak{g}^* \simeq \mathfrak{h}^* \times \mathfrak{k}^*$. Then there exists a natural action of K on $\mathcal{A}_{\hbar} //_{\eta} H$ with an induced quantum comoment map $\Phi_K: \mathfrak{k} \rightarrow \mathcal{A}_{\hbar} //_{\eta} H$, and there is a natural homomorphism of algebras

$$(\mathcal{A}_{\hbar} //_{\eta} H) //_{\kappa} K \rightarrow \mathcal{A}_{\hbar} //_{\gamma} G.$$

Proof. We first show that the reduced spaces are properly defined and there exists an action of K on $\mathcal{A}_{\hbar} //_{\eta} H$ with a quantum comoment map denoted Φ_K . First note that restricting the action of G yields an action of H on \mathcal{A}_{\hbar} , and the restriction $\Phi_H := \Phi|_{\mathfrak{h}}: \mathfrak{h} \rightarrow \mathcal{A}_{\hbar}$ is H -equivariant due to G -equivariance and the normality of H . Further, η is fixed by G and hence by H , so the reduction $\mathcal{A}_{\hbar} //_{\eta} H$ is well-defined.

We next define the action of K on $\mathcal{A}_{\hbar} //_{\eta} H$. Recall that

$$(2) \quad \mathcal{A}_{\hbar} //_{\eta} H = \{a + \mathcal{A}_{\hbar} \Phi_H(\mathcal{I}_{\eta}) : \text{Ad}_h(a) \in a + \mathcal{A}_{\hbar} \Phi_H(\mathcal{I}_{\eta}) \ \forall h \in H\}.$$

Define the action of K by $\text{Ad}_k(a + \mathcal{A}_{\hbar} \Phi_H(\mathcal{I}_{\eta})) := \text{Ad}_k(a) + \mathcal{A}_{\hbar} \Phi_H(\mathcal{I}_{\eta})$. This is independent of the choice of representative, as can be seen by the following calculation, taking $k \in K$ and $x \in \mathfrak{h}$:

$$\begin{aligned} \text{Ad}_k \Phi(x - \eta(x)\hbar) &= \Phi(\text{Ad}_k x - \eta(x)\hbar) && \Phi \text{ is } G\text{- and so } K\text{-equivariant} \\ &= \Phi(\text{Ad}_k x - \text{Ad}_k^*(\eta)(x)\hbar) && \eta \in (\mathfrak{h}^*)^G \subseteq (\mathfrak{h}^*)^K \\ &= \Phi(\text{Ad}_k x - \eta(\text{Ad}_k x)\hbar) \in \Phi(\mathcal{I}_{\eta}) && H \trianglelefteq G \text{ and so } \text{Ad}_k x \in \mathfrak{h} \end{aligned}$$

That $\text{Ad}_k(a + \mathcal{A}_{\hbar} \Phi_H(\mathcal{I}_{\eta}))$ remains H -invariant again follows from the normality of H , as $\text{Ad}_h \text{Ad}_k(\bar{a}) = \text{Ad}_{hk}(\bar{a}) = \text{Ad}_{kh'}(\bar{a}) = \text{Ad}_k \text{Ad}_{h'}(\bar{a}) = \text{Ad}_k(\bar{a})$.

Lastly, we need to exhibit a quantum comoment map $\Phi_K: \mathfrak{k} \rightarrow \mathcal{A}_{\hbar} //_{\eta} H$. We first define an η -twisted quantum comoment map, extending η by zero on \mathfrak{k} and defining

$$\begin{aligned} \Phi_{\eta}: \mathfrak{g} &= \mathfrak{h} \rtimes \mathfrak{k} \rightarrow \mathcal{A}_{\hbar} \\ \Phi_{\eta}(x) &:= \Phi(x) - \eta(x)\hbar \end{aligned}$$

To define the quantum comoment map Φ_K , consider \mathfrak{k} as the quotient $\mathfrak{g}/\mathfrak{h}$, choose an arbitrary lift from \mathfrak{k} to \mathfrak{g} , and apply the function $\pi_H \circ \Phi_{\eta}$, where π_H is the projection $\pi_H: \mathcal{A}_{\hbar} \rightarrow \mathcal{A}_{\hbar}/\mathcal{A}_{\hbar} \Phi_H(\mathcal{I}_{\eta})$. To see this is well-defined, note that for any $y \in \mathfrak{k}$ and $x \in \mathfrak{h}$,

$$\Phi_K(y) = \Phi_{\eta}(y + x) = \Phi(y) + \Phi(x - \eta(x)\hbar) \in \Phi(y) + \mathcal{A}_{\hbar} \Phi_H(\mathcal{I}_{\eta}).$$

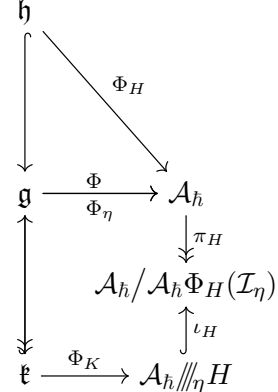
To see that the image of Φ_K lies within H -invariants, note that the semidirect product structure of \mathfrak{g} guarantees that for $y \in \mathfrak{k}$ and $h \in H$ there exists an $x \in \mathfrak{h}$ such that $\text{Ad}_h y = y + x$. From this we can see that

$$\text{Ad}_h \Phi_K(y) = \Phi_{\eta}(\text{Ad}_h y) = \Phi(y) + \Phi(x - \eta(x)\hbar) \in \Phi_K(y) + \mathcal{A}_{\hbar} \Phi_H(\mathcal{I}_{\eta}).$$

That this map is K -equivariant follows directly from the G -equivariance of Φ , and the quantum comoment condition $\frac{1}{\hbar}[\Phi_K(\xi), \cdot] = \xi_{\mathcal{A}_{\hbar} //_{\eta} H}$ follows from the corresponding condition for Φ and the above calculations showing the action of K is well-defined on $\mathcal{A}_{\hbar} //_{\eta} H$. Since $\kappa \in \mathfrak{k}^*$ is fixed by the action of G , and hence by K , we can therefore talk sensibly about the two-stage reduction $(\mathcal{A}_{\hbar} //_{\eta} H) //_{\kappa} K$. Furthermore, the action of K on $\mathcal{A}_{\hbar} //_{\eta} H$ descends to an action on $(\mathcal{A}_{\hbar} //_{\eta} H) / (\mathcal{A}_{\hbar} //_{\eta} H) \Phi_K(\mathcal{I}_{\kappa})$.

It remains to show that there is a map from the two-stage reduction $(\mathcal{A}_{\hbar} //_{\eta} H) //_{\kappa} K$ to the one-shot reduction $\mathcal{A}_{\hbar} //_{\gamma} G$. To this end we shall construct the map φ from fig. 3. Consider the maps

$$\begin{aligned} \tilde{\varphi}: (\mathcal{A}_{\hbar} //_{\eta} H) / (\mathcal{A}_{\hbar} //_{\eta} H) \Phi_K(\mathcal{I}_{\kappa}) &\rightarrow \mathcal{A}_{\hbar} / \mathcal{A}_{\hbar} \Phi(\mathcal{I}_{\gamma}) \\ \varphi: (\mathcal{A}_{\hbar} //_{\eta} H) //_{\kappa} K &\rightarrow \mathcal{A}_{\hbar} //_{\gamma} G, \end{aligned}$$



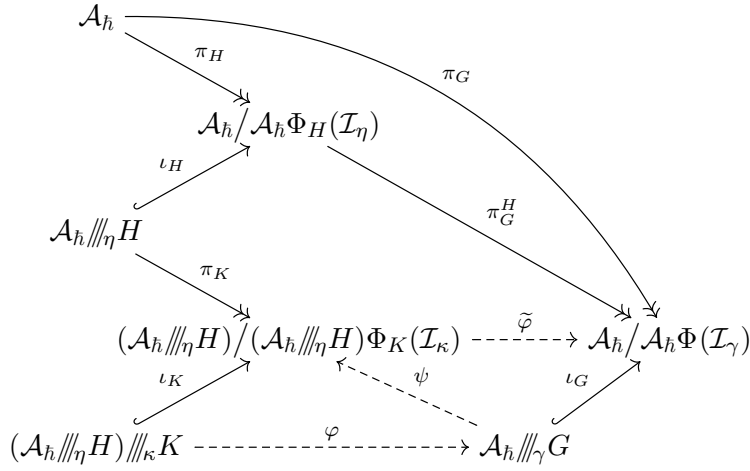


FIGURE 3. Quantum Hamiltonian reduction by stages

where $\tilde{\varphi}$ is defined by first lifting to $\mathcal{A}_h//_\eta H$ and then applying $\pi_G^H \circ \iota_H$, and φ is defined as the composition $\tilde{\varphi} \circ \iota_K$. To show these are well-defined requires checking that $\tilde{\varphi}$ doesn't depend on the lift chosen, and that the image of φ lies in G -invariants. These can both be checked by careful but straightforward calculations. We have therefore constructed a homomorphism φ from the two-stage to the one-shot reduction; it is merely the identity map suitably interpreted in the appropriate cosets:

$$\varphi((a + \mathcal{A}_h\Phi_H(\mathcal{I}_\eta)) + (\mathcal{A}_h//_\eta H)\Phi_K(\mathcal{I}_\kappa)) = a + \mathcal{A}_h\Phi(\mathcal{I}_\gamma). \quad \square$$

Corollary 2.7. *If this homomorphism φ induces an isomorphism $\overline{\varphi}: (A//_\eta H)//_\kappa K \xrightarrow{\sim} A//_\gamma G$ of the corresponding Poisson algebras, then it is itself an isomorphism. In particular, this holds if \mathcal{A}_h is a deformation quantisation of a Poisson manifold for which the classical reduction by stages hypotheses hold.*

Proof. The first statement follows from the fact that φ induces a homomorphism of Poisson algebras, and from the fact that A and \mathcal{A} have identical underlying vector spaces. That this induces an isomorphism if \mathcal{A}_h is a deformation quantisation of a Poisson manifold is just the classical Hamiltonian reduction by stages construction, which can be found in e.g. [MMO⁺, §5.3]. \square

Example 2.8. Consider the algebra $U_h(\mathfrak{sl}_3)$, which is a deformation quantisation of the Poisson variety \mathfrak{sl}_3^* . By restriction of the adjoint action of SL_3 , $U_h(\mathfrak{sl}_3)$ is acted on by the following groups:

$$N = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ t & s & 1 \end{pmatrix} : r, s, t \in \mathbb{C} \right\}, \quad M = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ t & 0 & 1 \end{pmatrix} : r, t \in \mathbb{C} \right\}, \quad K = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & s & 1 \end{pmatrix} : s \in \mathbb{C} \right\}.$$

Note that $N = M \rtimes K$, and the quantum comoment maps associated to the actions are given by the inclusions of their respective Lie algebras into $U_h(\mathfrak{sl}_3)$. Let $\chi \in \mathfrak{n}^*$ be the character corresponding to the regular nilpotent element $e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ under the Killing isomorphism, and let $\eta \in \mathfrak{m}^*$ and $\kappa \in \mathfrak{k}^*$ be the restrictions of χ .

The quantum Hamiltonian reduction $U_h(\mathfrak{sl}_3)//_\chi N$ is the polynomial ring $\mathbb{C}[z_1, z_2]$, where

$$z_1 := h_1^2 + h_1 h_2 + h_2^2 + 3\hbar(e_1 + e_2) \\ z_2 := 2h_1^3 + 3h_1^2 h_2 - 3h_1 h_2^2 - 2h_2^3 + 9\hbar e_1(h_1 + 2h_2) - 9\hbar e_2(2h_1 + h_2) + 27\hbar^2(e_3 + e_2).$$

These can be lifted to invariants under the action of M in $U_{\hbar}(\mathfrak{sl}_3)/U_{\hbar}(\mathfrak{sl}_3)\Phi_M(\mathcal{I}_{\eta})$ as

$$z_1 \mapsto z_1 + 3e_2(f_2 - \hbar) \quad z_2 \mapsto z_2 + 9(e_2h_2 + \hbar e_3 - \hbar e_2)(f_2 - \hbar).$$

Passing to the quotient $(U_{\hbar}(\mathfrak{sl}_3) //_{\eta} M) / (U_{\hbar}(\mathfrak{sl}_3) //_{\eta} M) \Phi_K(\mathcal{I}_{\kappa})$ yields the well-defined map ψ of fig. 3, and its image lies in K -invariants.

3. REDUCTION BY STAGES FOR W -ALGEBRAS IN TYPE A

Given the framework of quantum Hamiltonian reduction by stages, we can now try to find an explicit realisation in the case of W -algebras, as in fig. 1. From this point we shall work over \mathbb{C} and in type A, assuming that $\mathfrak{g} = \mathfrak{sl}_n$. In this case, we have a simple classification of both the conjugacy classes of nilpotent elements and of their good gradings.

Recall the set of nilpotent orbits in \mathfrak{sl}_n is parameterised by partitions of n , corresponding to the sizes of the Jordan blocks for the nilpotent. The set of nilpotent orbits in a Lie algebra always has a natural partial ordering, where $\mathcal{O}_{e_1} \leq \mathcal{O}_{e_2}$ is defined to mean $\mathcal{O}_{e_1} \subseteq \overline{\mathcal{O}_{e_2}}$. In type A, this coincides with the dominance ordering on partitions: take $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_k)$, where λ and μ are padded on the right with zeros if necessary, and define $\lambda \leq \mu$ to mean that $\sum_{i=1}^{\ell} \lambda_i \leq \sum_{i=1}^{\ell} \mu_i$ for every $\ell = 1, \dots, k$. A classical theorem of Gerstenhaber classifies the covering relations in the dominance ordering, and roughly corresponds to ‘sliding a box up’ in the corresponding Young diagram.

Lemma 3.1. [Gerstenhaber] *The partition λ covers μ if and only if there exist indices $j < k$ with $\mu_j = \lambda_j - 1$, $\mu_k = \lambda_k + 1$ and $\lambda_i = \mu_i$ otherwise, where j is the smallest index such that $0 \leq \lambda_k < \lambda_j - 1$ and either $k = j + 1$ or $\lambda_k = \lambda_j - 2$.*

Example 3.2. The partitions $\lambda = (3)$, $\mu = (2, 1)$ and $\nu = (1, 1, 1)$ cover one another in turn.

$$\lambda = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \quad \mu = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \quad \nu = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

3.1. Pyramids. The problem of classifying all good gradings has been solved by Elashvili and Kac [EK]; in the classical types, this is accomplished using a combinatorial structure known as a *pyramid*. In type A, pyramids are an enriched version of Young diagrams, allowing horizontal shifts of the rows according to certain conditions. In this paper we shall use the French convention for Young diagrams.

Definition 3.3. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n . A *pyramid of shape λ* is a Young diagram of shape λ consisting of boxes of size 2, along with integer horizontal row shifts such that the co-ordinates of the first (resp. last) boxes in each row form an increasing (resp. decreasing) sequence.

A *filling* of a pyramid is a labelling of each of the boxes with a number between 1 and n , such that there are no repeated labels. Given a filled pyramid, the column and row of the box labelled k are denoted $\text{col}(k)$ and $\text{row}(k)$, respectively. We say ℓ is *right-adjacent* to k , denoted $k \rightarrow \ell$, if $\text{row}(k) = \text{row}(\ell)$ and $\text{col}(k) + 2 = \text{col}(\ell)$.

Note. The row and column of a box are only well-defined up to an integer shift. However, since we’ll only ever be concerned with differences of row and column numbers, this will not cause a problem.

When filling pyramids, we shall most often choose the labelling so that it increases first up columns and then left to right.

Example 3.4. The three pyramids of shape $(4, 3)$ follow, each with a sample filling.

$$\begin{array}{c}
 \text{(3a)} \quad \begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline 1 & 3 & 5 \\ \hline \end{array} \quad \text{(3b)} \quad \begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline 1 & 3 & 5 \\ \hline \end{array} \quad \text{(3c)} \quad \begin{array}{|c|c|c|} \hline 3 & 5 & 7 \\ \hline 1 & 2 & 4 \\ \hline \end{array}
 \end{array}$$

Theorem 3.5. [EK, Theorem 4.2] *There is a bijection between the pyramids of size n and the set of good gradings in \mathfrak{sl}_n up to conjugacy. The same holds in \mathfrak{gl}_n .*

Consider a filled pyramid P . The nilpotent element e_P associated to P is just the nilpotent element associated to P considered as a Young tableau, namely $\sum_{i \rightarrow j} E_{ij}$. The grading Γ_P associated to P is defined by declaring E_{ij} to be of graded degree $\text{col}(j) - \text{col}(i)$. It can be checked that this grading is good for e_P .

3.2. Reduction by stages for W-algebras. In this section we shall use the machinery of pyramids to produce a quantum Hamiltonian reduction by stages for W-algebras in type A. Since our reductions are by nilpotent groups, it will suffice to work with the Lie algebras, which completely determine the actions of the corresponding algebraic groups.

Objective 3.6. *Let $\mathfrak{g} = \mathfrak{sl}_n$, and $e_1, e_2 \in \mathfrak{g}$ be two nilpotent elements such that $\mathcal{O}_{e_1} < \mathcal{O}_{e_2}$. We would like to construct an algebraic group K with a quantum Hamiltonian action on $U(\mathfrak{g}, e_1)$, along with a character $\kappa \in \mathfrak{k}^*$, such that $U(\mathfrak{g}, e_2) \simeq U(\mathfrak{g}, e_1) //_{\kappa} K$.*

It will suffice to produce such a construction for every pair such that \mathcal{O}_2 covers \mathcal{O}_1 . For any such pair we will chose nilpotent elements $e_i \in \mathcal{O}_i$ with respective duals $\chi_i \in \mathfrak{g}^*$ for $i = 1, 2$, a good grading Γ_1 for e_1 with a Premet subalgebra \mathfrak{m}_1 , and a subalgebra $\mathfrak{m}_2 \supseteq \mathfrak{m}_1$ satisfying:

- SR1. \mathfrak{m}_2 decomposes as a semidirect product $\mathfrak{m}_2 = \mathfrak{m}_1 \rtimes \mathfrak{k}$.
- SR2. χ_2 restricts to a character of \mathfrak{m}_2 , and $\chi_2 = (\chi_1, \kappa)$ in the above decomposition.
- SR3. the subalgebra \mathfrak{k} annihilates χ_1 .

Since \mathfrak{m}_1 is a Premet subalgebra for e_1 , the corresponding quantum Hamiltonian reduction by stages will ensure that

$$U(\mathfrak{g}) //_{\chi_2} \mathfrak{m}_2 \simeq (U(\mathfrak{g}) //_{\chi_1} \mathfrak{m}_1) //_{\kappa} \mathfrak{k} = U(\mathfrak{g}, e_1) //_{\kappa} \mathfrak{k}.$$

We will therefore provide a construction satisfying these conditions, and conjecture that the algebra $U(\mathfrak{g}) //_{\chi_2} \mathfrak{m}_2$ is isomorphic to the W-algebra $U(\mathfrak{g}, e_2)$. Provided this conjecture holds, this will accomplish objective 3.6.

3.2.1. The general construction. Let μ be the partition corresponding to the nilpotent e_1 . We will construct a right-aligned pyramid for μ , i.e. a pyramid for which the rightmost boxes in each row all lie in the same column, and number the boxes from bottom to top and left to right. This determines an even good grading Γ_1 and Premet subalgebra \mathfrak{m}_1 for e_1 . By lemma 3.1, for every partition λ which covers μ in the dominance ordering, there is a pair of integers $i < j$ for which λ is obtained from μ by ‘sliding a box down’ from row j to row i . Define e_2 as

$$(4) \quad e_2 := e_1 + \sum_{\substack{\text{row}(k)=i, \text{row}(\ell)=j \\ \text{col}(k)=\text{col}(\ell)}} E_{k\ell},$$

and define the Lie algebras \mathfrak{m}_2 and \mathfrak{k} by

$$(5) \quad \mathfrak{k} := \langle E_m \rangle_{m=1}^{j-i} \quad \text{and} \quad \mathfrak{m}_2 := \mathfrak{m}_1 + \mathfrak{k}, \quad \text{where} \quad E_m := \sum_{\substack{i \leq \text{row}(k) < \text{row}(\ell) \leq j \\ \text{row}(\ell) - \text{row}(k) = m \\ \text{col}(k) = \text{col}(\ell)}} E_{k\ell}.$$

Let us further define a semisimple element h'_2 :

$$(6) \quad h'_2 := \sum_{\substack{\text{row}(\ell)=s, s \neq i, j \\ t=0, \dots, \lambda_s-1 \\ \ell \text{ is } t\text{-th from the left}}} (\lambda_s - 1 - 2t)E_{\ell\ell} + \sum_{\substack{\text{row}(\ell)=i, \text{row}(m)=j \\ t=0, \dots, \lambda_i \\ \text{col}(m)=\text{col}(\ell)-2 \\ \ell \text{ is } t\text{-th from the left}}} (\lambda_i + K - 2t)(E_{\ell\ell} + E_{mm}).$$

In the second term, the E_{mm} term is omitted if there is no m satisfying the conditions for the given t , and $E_{\ell\ell}$ is omitted for $t = \lambda_i$. Here, K is the unique constant so that h'_2 has trace zero. Note that h'_2 is a semisimple element for which $[h'_2, e_2] = 2e_2$; we shall show that h'_2 determines a good grading for e_2 in lemma 3.12, however \mathfrak{m}_2 is not in general a Premet subalgebra for this grading, nor does there necessarily exist an \mathfrak{sl}_2 -triple containing e_2 and h'_2 .

Remark 3.7. Note that \mathfrak{k} is an abelian Lie algebra, and that

$$[\mathfrak{m}_2, \mathfrak{m}_2] = [\mathfrak{m}_2, \mathfrak{m}_1] \subseteq \left[\bigoplus_{k \leq 0} \mathfrak{g}_k, \bigoplus_{\ell \leq -2} \mathfrak{g}_\ell \right] \subseteq \bigoplus_{k \leq -2} \mathfrak{g}_k = \mathfrak{m}_1 \subseteq \mathfrak{m}_2.$$

This confirms that \mathfrak{m}_2 is closed under the Lie bracket, and further that \mathfrak{m}_1 is an ideal in \mathfrak{m}_2 ; hence, \mathfrak{m}_2 is a semi-direct product $\mathfrak{m}_1 \rtimes \mathfrak{k}$.

Example 3.8. Let $\mathfrak{g} = \mathfrak{sl}_6$ and consider $\mu = (2, 2, 2)$. The right-aligned pyramid P_1 , nilpotent element e_1 and Premet subalgebra \mathfrak{m}_1 are as follows:

$$P_1 = \begin{array}{|c|c|} \hline 3 & 6 \\ \hline 2 & 5 \\ \hline 1 & 4 \\ \hline \end{array} \quad e_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathfrak{m}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \end{pmatrix}$$

The unique covering partition is $\lambda = (3, 2, 1)$, which is obtained by ‘sliding a box from row 3 to row 1’. Applying the above procedure with $i = 1$ and $j = 3$ results in

$$\begin{aligned} e_2 &= e_1 + E_{13} + E_{46} & \mathfrak{m}_2 &= \mathfrak{m}_1 + \langle E_1, E_2 \rangle \\ &= \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} & &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & a & 0 & 0 \\ * & * & * & b & a & 0 \end{pmatrix} \end{aligned}$$

where $E_1 = E_{21} + E_{32} + E_{54} + E_{65}$ and $E_2 = E_{31} + E_{64}$. Further,

$$h'_2 = (E_{22} - E_{55}) + (2E_{11} + 0E_{44} + 0E_{33} - 2E_{66})$$

3.2.2. Properties of the construction.

Theorem 3.9. Under the above circumstances, e_2 is a nilpotent element of type λ , \mathfrak{m}_2 is a Lie algebra and conditions **SR1** to **SR3** hold. Consequently, theorem 2.6 holds, and so there is homomorphism from the quantum Hamiltonian reduction by stages to the one-shot reduction:

$$(U(\mathfrak{g}) //_{\chi_1} \mathfrak{m}_1) //_{\kappa} \mathfrak{k} = U(\mathfrak{g}, e_1) //_{\kappa} \mathfrak{k} \rightarrow U(\mathfrak{g}) //_{\chi_2} \mathfrak{m}_2.$$

We shall prove in theorem 4.4 that this homomorphism is, in fact, an isomorphism, but will leave this discussion until the necessary framework has been developed. Before proving the theorem, we should introduce a result of [EK]: given any filled pyramid P with corresponding nilpotent element e , the centraliser $\mathfrak{z}_{\mathfrak{g}}(e)$ can be read

off the pyramid P . Let $\mu = (\mu_1, \dots, \mu_k)$, so the i th row of the pyramid has μ_i boxes, and let $b_{i,j}$ be the standard basis vector corresponding to the index of the box in the i th row, j th from the right in the filled pyramid. We can represent an endomorphism of \mathbb{C}^n by specifying where each of the basis vectors $b_{i,j}$ is sent in an arrow diagram.

Elashvili and Kac define a collection of endomorphisms in \mathfrak{gl}_n , denoted $E_i^j[r]$, where i and j range over the rows of the pyramid and $r \in \mathbb{N}$ varies over a range depending on μ_i and μ_j . These endomorphisms are defined in fig. 4, where any basis vector not specified is sent to zero.

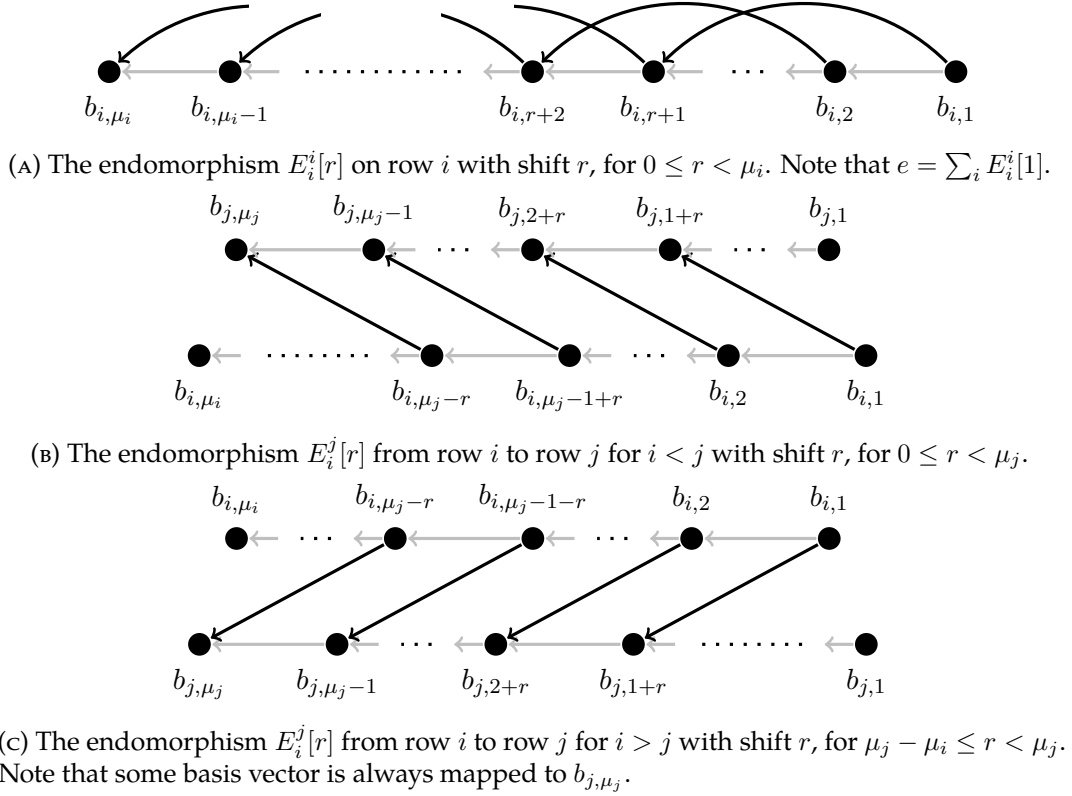


FIGURE 4. Endomorphisms of \mathbb{C}^n commuting with e .

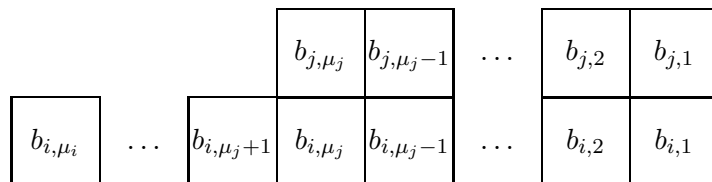
Note. The nilpotent e is shown in grey in each row for reference.

Lemma 3.10. [EK] Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition of n , and consider a filled pyramid of shape μ with associated nilpotent e . Then the collection $\{E_i^j[r]\}$, where

$$i, j \in \{1, \dots, k\} \quad \text{and} \quad \begin{aligned} 0 \leq r < \mu_j & \quad \text{if } i \leq j \\ \mu_j - \mu_i \leq r < \mu_j & \quad \text{if } i > j \end{aligned}$$

forms a basis of the centraliser $\mathfrak{z}_{\mathfrak{gl}_n}(e)$, and those which lie in \mathfrak{sl}_n form a basis of $\mathfrak{z}_{\mathfrak{sl}_n}(e)$.

Proof of theorem 3.9. To prove that e_2 has the correct Jordan type, it suffices to exhibit a Jordan basis. Note that a Jordan basis can be read off the rows of the pyramid, proceeding from right to left.



The Jordan basis for e_1 in row i of this pyramid is therefore

$$b_{i,\mu_i} \leftarrow \cdots \leftarrow b_{i,\mu_j+1} \leftarrow b_{i,\mu_j} \leftarrow b_{i,\mu_j-1} \leftarrow \cdots \leftarrow b_{i,2} \leftarrow b_{i,1}.$$

The Jordan basis for e_2 is identical to that of e_1 except for those strings corresponding to rows i and j . The Jordan basis in those rows is

$$\begin{aligned} \mu_j b_{i,\mu_i} \leftarrow \cdots \leftarrow \mu_j b_{i,\mu_j} \leftarrow ((\mu_j - 1)b_{i,\mu_j-1} + b_{j,\mu_j}) \leftarrow \cdots \leftarrow (kb_{i,k} + b_{j,k+1}) \leftarrow \cdots \leftarrow b_{j,1}, \\ (b_{i,\mu_j-1} - b_{j,\mu_j}) \leftarrow \cdots \leftarrow ((\mu_j - k)b_{i,k} - b_{j,k+1}) \leftarrow \cdots \leftarrow ((\mu_j - 1)b_{i,1} - b_{j,2}), \end{aligned}$$

of lengths $\mu_i + 1 = \lambda_i$ and $\mu_j - 1 = \lambda_j$, respectively.

Condition **SR1** is shown in remark 3.7. To check **SR2**, note that $\chi_2|_{\mathfrak{m}_1} = \chi_1$ by construction, and so $\chi_2 = (\chi_1, \kappa)$ for some $\kappa \in \mathfrak{k}^*$. To confirm that χ_2 is a character of \mathfrak{m}_2 , recall from remark 3.7 that $[\mathfrak{m}_2, \mathfrak{m}_2] = [\mathfrak{m}_1 + \mathfrak{k}, \mathfrak{m}_1]$. However, since $\chi_2([\mathfrak{m}_1, \mathfrak{m}_1]) = \chi_1([\mathfrak{m}_1, \mathfrak{m}_1]) = 0$, it remains only to check that $\chi_2([\mathfrak{k}, \mathfrak{m}_1]) = 0$. We shall check this on the generating set $\{[E_m, E_{\ell k}] : 1 \leq m \leq j - i, \text{col}(k) < \text{col}(\ell)\}$.

$$\chi_2([E_m, E_{\ell k}]) = \langle e_2, [E_m, E_{\ell k}] \rangle = \langle [e_2, E_m], E_{\ell k} \rangle$$

Using the language of fig. 4, note that

$$(7) \quad [e_2, E_m] = \left[e_1 + E_j^i[0], \sum_{i \leq s \leq j-m} E_s^{s+m}[0] \right] = \sum_{i \leq s \leq j-m} \left[E_j^i[0], E_s^{s+m}[0] \right] \in \mathfrak{g}_0.$$

Here the second equality follows from the fact that $E_s^{s+m}[0]$ commutes with e_1 ; that χ_2 annihilates eq. (7) now follows from property **GG5**, and the fact that $E_{\ell k} \in \bigoplus_{t < 0} \mathfrak{g}_t$. This further establishes the claim that \mathfrak{k} annihilates both χ_2 and χ_1 : hence condition **SR3** also holds. This completes the proof of theorem 3.9. \square

Theorem 3.11. *The pair e_2 and \mathfrak{m}_2 satisfy properties $\chi 1$ to $\chi 4$.*

Proof. Property $\chi 1$ is manifest from the construction, and property $\chi 4$ is a subclaim of condition **SR2**. Property $\chi 2$ follows from the fact that e_1 itself satisfies it, along with an application of the orbit–stabiliser theorem and lemma 3.10.

We prove property $\chi 3$ by directly calculating $\mathfrak{m}_2 \cap \mathfrak{z}_{\mathfrak{g}}(e_2)$. In the coming calculation, we use the following conventions:

- Recall that $k \rightarrow \ell$ means that ℓ is right-adjacent to k .
- If $\text{row}(k) = i$, then $k \uparrow^p$ indicates that p is the box such that $\text{row}(p) = j$ and $\text{col}(p) = \text{col}(k)$, if such exists. Similarly, if $\text{row}(s) = j$, then $q \uparrow^s$ indicates that q is such that $\text{row}(q) = i$ and $\text{col}(q) = \text{col}(s)$, if such exists.
- $A_{k\ell} = 0$ if there do not exist k and ℓ which satisfy the adjacency relations specified below, and $B_m = 0$ if $m < 1$ or $m > j - i$.

Taking the commutator of e_2 with a generic element of \mathfrak{m}_2 results in the following:

$$\begin{aligned} (8) \quad & \left[e_2, \sum_{\text{col}(v) < \text{col}(u)} A_{uv} E_{uv} + \sum_{m=1}^{j-i} B_m E_m \right] = \\ & = \sum_{\text{col}(s) < \text{col}(k)} \left(A_{\ell s} - A_{kr} + \begin{cases} A_{ps} & \text{row}(k) = i, k \uparrow^p \\ 0 & \text{otherwise} \end{cases} - \begin{cases} A_{kq} & \text{row}(s) = j, q \uparrow^s \\ 0 & \text{otherwise} \end{cases} \right) E_{ks} \\ & + \sum_{\text{col}(s) = \text{col}(k)} \left(A_{\ell s} - A_{kr} + \begin{cases} B_m & \text{row}(k) = i, \text{row}(s) = j - m \\ -B_m & \text{row}(s) = j, \text{row}(k) = i + m \\ 0 & \text{otherwise} \end{cases} \right) E_{ks}. \end{aligned}$$

For eq. (8) to vanish, we will prove that all of A_{uv} and B_m must vanish as well.

Case 1 ($A_{uw} = 0$ for $\text{col}(v) < \text{col}(u)$, $\text{row}(u) \neq i$ and $\text{row}(v) \neq j$).

Examining the coefficient of E_{uw} for $v \rightarrow w$ yields $A_{tw} - A_{uw}$, where $u \rightarrow t$. We can prove the claim by induction on the distance of u from the right of the pyramid. The base case is when u is rightmost in its row: then $A_{tw} = 0$, and so $A_{uw} = 0$. The same argument assuming the inductive hypothesis for all u within n boxes of the right proves the claim for all u within $n + 1$ boxes of the right, completing the induction.

Case 2 ($A_{uw} = 0$ for $\text{col}(v) < \text{col}(u) - 1$, $\text{row}(u) = i$ and $\text{row}(v) \neq j$).

Examining the coefficient of E_{uw} for $v \rightarrow w$ yields $A_{tw} - A_{uw} + A_{pw}$, where $u \rightarrow t$ and $u \uparrow^p$. But $A_{pw} = 0$ by case 1, and so the same argument as above completes the case.

Case 3 ($A_{uw} = 0$ for $\text{col}(v) < \text{col}(u) - 1$, $\text{row}(u) \neq i$ and $\text{row}(v) = j$).

Apply the argument of case 2 *mutatis mutandis*.

Case 4 ($A_{uw} = 0$ for $\text{col}(v) < \text{col}(u) - 1$, $\text{row}(u) = i$ and $\text{row}(v) = j$).

The conclusions of cases 2 and 3 allow the same argument to again be applied *mutatis mutandis*.

Case 5 ($A_{uw} = 0$ for $\text{col}(v) = \text{col}(u) - 1$).

If neither $\text{row}(u) = i$ or $\text{row}(v) = j$ this is dealt with by case 1, while if both of these hold there is no contribution from any B_m . Since the arguments are symmetric, we'll assume that $\text{row}(u) = i$.

Assume that $\text{row}(v) = j - m$ for $1 \leq m \leq j - i$; otherwise there is no contribution from any B_m and we're done. Since e_2 covers e_1 there are exactly the same number of boxes in the two rows; let the boxes of row i be labelled from the left u_1, \dots, u_k and the boxes of row j be labelled v_1, \dots, v_k . The sum of the co-efficients of $E_{u_1 v_1}$ up to $E_{u_k v_k}$ is kB_m , which proves that $B_m = 0$. The argument of case 1 proves that the remaining A_{uw} must vanish, which completes this last case and the proof of the theorem. \square

Finally, recalling the element h'_2 from eq. (6), we establish the following result for future reference.

Lemma 3.12. *The grading coming from the semisimple element h'_2 is good for e_2 . Further, $\text{ad } h'_2$ preserves \mathfrak{m}_1 .*

Proof. By construction we have that $[h'_2, e_2] = 2e_2$, so all that remains is to show that $\bigoplus_{i < 0} \mathfrak{g}_i \cap \mathfrak{z}_{\mathfrak{g}}(e_2) = \{0\}$, which is equivalent to property GG2. Considering the basis of the centraliser $\mathfrak{z}_{\mathfrak{g}}(e_1)$ given in lemma 3.10, note that the basis of the centraliser $\mathfrak{z}_{\mathfrak{g}}(e_2)$ is closely related: it differs only in that endomorphisms which involve basis vectors in rows i and j have these replaced by an appropriate linear combination of basis vectors as in the proof of lemma 3.10. However, these linear combinations lie in the zero weight space of h'_2 by construction; this proves that any element of the centraliser cannot lie in strictly negative degree. That $\text{ad } h'_2$ preserves \mathfrak{m}_1 follows immediately from the fact that h'_2 is diagonal. \square

3.3. Relation to W-algebras.

Conjecture 3.13. *For nilpotents $e_1, e_2 \in \mathfrak{g}$ and subalgebras $\mathfrak{m}_1, \mathfrak{m}_2 \subseteq \mathfrak{g}$ as defined in eqs. (4) and (5), the reduction by stages $U(\mathfrak{g}, e_1) //_{\kappa} \mathfrak{k} \simeq U(\mathfrak{g}) //_{\chi_2} \mathfrak{m}_2$ is isomorphic to the W-algebra $U(\mathfrak{g}, e_2)$.*

Remark 3.14. This conjecture is a special case of a more general conjecture due to Premet, based on his work in [Pre]. Specifically, Premet conjectures that for any

pair of subalgebra \mathfrak{m} and nilpotent e which satisfy properties $\chi 1$ to $\chi 4$, the quantum Hamiltonian reduction $U(\mathfrak{g}) //_{\chi} \mathfrak{m}$ is isomorphic to the W-algebra $U(\mathfrak{g}, e)$. In fact, Premet has proven this conjecture in the case that the base field is of non-zero characteristic p .

Proposition 3.15. *Conjecture 3.13 holds for e_1 a subregular and e_2 a regular nilpotent.*

Proof. The subalgebra \mathfrak{m}_2 constructed is simply the maximal nilpotent subalgebra of lower-triangular matrices \mathfrak{n}^- . This is a Premet subalgebra for e_2 . \square

Remark 3.16. The construction detailed in this section can be modified slightly to give a stronger version of proposition 3.15. Instead of choosing a right-aligned pyramid of shape μ , one can choose a pyramid which is right-aligned but for a leftward shift of 1 at row i and another leftward shift of 1 at row $j + 1$. This necessitates a choice of Lagrangian $\mathfrak{l} \subseteq \mathfrak{g}_{-1}$; this choice can be made so that the resulting Premet subalgebra can be extended to a Premet subalgebra for a pyramid of shape λ , which is right-aligned but for a leftward shift of 1 at row $i + 1$ and another leftward shift of 2 at row j .

$$\begin{array}{c}
 \begin{array}{|c|} \hline 6 \\ \hline \end{array} \\
 \begin{array}{|c|c|} \hline 4 & 9 \\ \hline \end{array} \\
 \begin{array}{|c|c|} \hline 3 & 8 \\ \hline \end{array} \\
 \begin{array}{|c|c|} \hline 2 & 7 \\ \hline \end{array} \\
 \begin{array}{|c|c|c|} \hline 1 & 5 & 10 \\ \hline \end{array}
 \end{array}
 \quad < \quad
 \begin{array}{c}
 \begin{array}{|c|} \hline 6 \\ \hline \end{array} \\
 \begin{array}{|c|} \hline 4 \\ \hline \end{array} \\
 \begin{array}{|c|c|} \hline 3 & 8 \\ \hline \end{array} \\
 \begin{array}{|c|c|c|} \hline 2 & 7 & 9 \\ \hline \end{array} \\
 \begin{array}{|c|c|c|} \hline 1 & 5 & 10 \\ \hline \end{array}
 \end{array}$$

$$\mu = (3, 2, 2, 2, 1) \qquad \lambda = (3, 3, 2, 1, 1)$$

For this new pyramid and compatible choice of Lagrangian, theorem 3.9 remains true. Furthermore, proposition 3.15 and its proof hold not only for e_1 a subregular and e_2 a regular nilpotent, but more generally for any pair of nilpotent elements e_1 and e_2 of types $\mu = (\mu_1, \dots, \mu_k, 1)$ and $\lambda = (\mu_1, \dots, \mu_k + 1)$, respectively.

Example 3.17. Consider \mathfrak{sl}_4 , and e_1 a nilpotent of type $(2, 2)$. This is covered by the subregular nilpotent e_2 , and so the construction will produce an algebra $U(\mathfrak{g}) //_{\chi_2} \mathfrak{m}_2$. The associated graded algebra of $U(\mathfrak{g}, e_2)$ is the ring of functions on the Slodowy slice \mathcal{S}_{χ_2} , and the associated graded algebra of the reduced space is $\mathbb{C}[\mathfrak{g}^* //_{\chi_2} M_2]$.

$$\begin{aligned}
 \mathcal{S}_{\chi_2} &= \left\{ \begin{pmatrix} a & 1 & 0 & 0 \\ b - 3a^2 & a & 1 & 0 \\ c + 20a^3 & b - 3a^2 & a & d \\ f & 0 & 0 & -3a \end{pmatrix} : a, b, c, d, f \in \mathbb{C} \right\} \\
 \mathfrak{g}^* //_{\chi_2} M_2 &\simeq \left\{ \begin{pmatrix} 0 & 1 & 1 & 0 \\ x + \frac{u+v}{4} & 0 & 0 & 1 \\ \frac{-3u+v}{4} & -2y & 0 & 1 \\ z + \frac{u+v}{2}y & \frac{u-3v}{4} & x + \frac{u+v}{4} & 0 \end{pmatrix} : u, v, x, y, z \in \mathbb{C} \right\} \\
 \mathbb{C}[\mathcal{S}_{\chi_2}] &= \mathbb{C}[a, b, c, d, f], \quad \mathbb{C}[\mathfrak{g}^* //_{\chi_2} M_2] = \mathbb{C}[u, v, x, y, z]
 \end{aligned}$$

These are isomorphic as Poisson algebras, as shall be shown below.

Consider the ring homomorphism $\varphi: \mathbb{C}[\mathcal{S}_{\chi_2}] \rightarrow \mathbb{C}[\mathfrak{g}^* //_{\chi_2} M_2]$ defined by

$$\varphi(a) = \frac{-1}{3}y \quad \varphi(b) = x \quad \varphi(c) = 2z - \frac{8}{3}xy \quad \varphi(d) = v + x + y^2 \quad \varphi(f) = -u - x - y^2.$$

The non-zero Poisson brackets are given by the formulae:

$$\begin{aligned} \{a, d\} &= \frac{-1}{24}d & \{c, d\} &= \frac{1}{6}bd & \{u, y\} &= \frac{1}{8}(u + x + y^2) & \{u, z\} &= \frac{1}{4}x(u + x + y^2) \\ \{a, f\} &= \frac{1}{24}f & \{c, f\} &= \frac{-1}{6}bf & \{v, y\} &= \frac{-1}{8}(v + x + y^2) & \{v, z\} &= \frac{-1}{4}x(v + x + y^2) \\ \{d, f\} &= \frac{-27}{2}a^3 + ab - \frac{1}{8}c & \{u, v\} &= \frac{-1}{4}(z + xy + 2(u + v)y) \end{aligned}$$

It can be checked that this map is a ring isomorphism and also preserves the Poisson bracket; it hence induces an isomorphism of the Poisson varieties $\mathcal{S}_{\chi_2} \simeq \mathfrak{g}^* //_{\chi_2} M_2$. Furthermore, this map preserves the characteristic polynomial.

Thus, we know that our algebra $U(\mathfrak{g}) //_{\chi_2} \mathfrak{m}_2$ is a deformation quantisation of the Slodowy slice \mathcal{S}_{χ_2} . Since there is, up to isomorphism, a unique such deformation quantisation which is G -equivariant, and the W-algebra $U(\mathfrak{g}, e_2)$ is such a deformation quantisation, it follows that $U(\mathfrak{g}) //_{\chi_2} \mathfrak{m}_2 \simeq U(\mathfrak{g}, e_1) //_{\kappa} \mathfrak{k}$ is isomorphic to $U(\mathfrak{g}, e_2)$.

4. THE REPRESENTATION THEORY OF W-ALGEBRAS

The construction of quantum Hamiltonian reduction by stages has a number of applications to the representation theory of W-algebras. Theorem 3.9 has an immediate corollary relating the categories of modules over $U(\mathfrak{g}, e_1)$ and $U(\mathfrak{g}, e_2)$.

Corollary 4.1. *Let e_1 and e_2 be two nilpotent elements of \mathfrak{sl}_n such that e_2 covers e_1 in the dominance ordering; then the quantum Hamiltonian reduction by stages construction produces an adjoint pair of functors $U(\mathfrak{g}, e_1)\text{-mod} \rightleftarrows (U(\mathfrak{g}) //_{\chi_2} \mathfrak{m}_2)\text{-mod}$. If conjecture 3.13 holds, then there exists an adjunction $U(\mathfrak{g}, e_1)\text{-mod} \rightleftarrows U(\mathfrak{g}, e_2)\text{-mod}$ for any pair of nilpotents $e_2 \geq e_1$.*

Proof. Note that the quotient $U(\mathfrak{g}, e_1)/U(\mathfrak{g}, e_1)\mathfrak{k}_{\kappa}$ is a $(U(\mathfrak{g}, e_1), U(\mathfrak{g}) //_{\chi_2} \mathfrak{m}_2)$ -bimodule, where the left module structure comes from left multiplication by $U(\mathfrak{g}, e_1)$ and the right module structure comes from the fact that $U(\mathfrak{g}) //_{\chi_2} \mathfrak{m}_2 \simeq (U(\mathfrak{g}, e_1)/\mathfrak{k}_{\kappa})^{\mathfrak{k}}$. This proves the existence of the first adjunction.

If conjecture 3.13 holds, then the latter algebra is isomorphic to $U(\mathfrak{g}, e_2)$. Since adjunctions can be composed, and their composition is itself an adjunction, we can form such an adjunction for any pair of nilpotents $e_2 \geq e_1$ by composing along a sequence of covering relations. \square

There are also applications to the W-algebraic analogue of the BGG category \mathcal{O} : a full subcategory of $U(\mathfrak{g}, e)\text{-mod}$ whose definition we shall recall in the next section, and which has been studied in, e.g. [BGK, Web]. In [Los3], Loseu investigates its structure and constructs an equivalence between it and a certain subcategory of Whittaker modules in $U(\mathfrak{g})\text{-mod}$. The objective of this section is to prove a similar equivalence in type A, relating the W-algebraic categories \mathcal{O} for different nilpotents to one another. In what follows we shall always assume conjecture 3.13 holds.

4.1. Categories \mathcal{O} and other related categories for W-algebras. To discuss the categories \mathcal{O} for the W-algebra $U(\mathfrak{g}, e)$, we need to fix a choice of parabolic subalgebra $\mathfrak{p} \subseteq \mathfrak{g}$ such that (e, h, f) is contained in the Levi subalgebra $\mathfrak{l} \subseteq \mathfrak{p}$. Further, we shall fix a maximal torus \mathfrak{t} of the centraliser $\mathfrak{z}_{\mathfrak{g}}(e)$, noting that $\mathfrak{t} \subseteq \mathfrak{l}$.

In place of the choice of parabolic, Loseu instead chooses a cocharacter θ of T viewed as an element of \mathfrak{t} ; this uniquely determines a parabolic as the positive eigenspaces of θ . Different choices of θ will only matter inasmuch as they determine different parabolics, so as far as we are concerned the two points of view are equivalent.

This choice of parabolic and maximal torus allows us to define a pre-order on the weights of \mathfrak{t} : $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a linear combination of the weights of \mathfrak{t} acting on \mathfrak{p} . The existence of an embedding $U(\mathfrak{t}) \hookrightarrow U(\mathfrak{g}, e)$ (cf. [BGK, Theorem 3.3])

allows any $U(\mathfrak{g}, e)$ -module to be decomposed into generalised weight spaces with respect to \mathfrak{t} .

Note also that, as proven by Premet, $Z(\mathfrak{g}, e) := Z(U(\mathfrak{g}, e))$ is isomorphic to the ordinary centre of the universal enveloping algebra $U(\mathfrak{g})$, and that the natural map $Z(\mathfrak{g}) \rightarrow U(\mathfrak{g}, e)$ is an isomorphism onto the centre. Thus, central characters of $U(\mathfrak{g})$ can be translated to central characters of $U(\mathfrak{g}, e)$.

This allows for a number of different full subcategories of $U(\mathfrak{g}, e)\text{-mod}$ to be defined, the objects of which satisfy various subsets of the following conditions.

- (O1) The \mathfrak{t} -weights are contained in a finite union of sets of the form $\{\mu : \mu \leq \lambda\}$.
- (O2) The generalised weight spaces with respect to \mathfrak{t} are finite-dimensional.
- (O3) The action of \mathfrak{t} on the module is semisimple.
- (O4) The action of $Z(\mathfrak{g}, e) := Z(U(\mathfrak{g}, e))$ on the module is semisimple.

The notation used for these categories differs amongst different papers. We will mostly keep to a pared-down version of the notation used in Webster's papers [Web], but since the machinery and proof of Loseu's work [Los3] is extremely important here, we shall present it as well. Loseu's notation leaves the nilpotent e implicit, which would render it ambiguous in the context of this paper.

Conditions	1	1,2	1,3	1,2,3	1,2,4
Notation	$\tilde{\mathcal{O}}(e, \mathfrak{p})$	$\hat{\mathcal{O}}(e, \mathfrak{p})$		$\mathcal{O}(e, \mathfrak{p})$	$\mathcal{O}'(e, \mathfrak{p})$
Webster		$\hat{\mathcal{O}}(\mathcal{W}_e, \mathfrak{p})$		$\mathcal{O}(\mathcal{W}_e, \mathfrak{p})$	$\mathcal{O}'(\mathcal{W}_e, \mathfrak{p})$
Loseu	$\tilde{\mathcal{O}}(\theta)$	$\mathcal{O}(\theta)$	$\tilde{\mathcal{O}}^{\mathfrak{t}}(\theta)$	$\mathcal{O}^{\mathfrak{t}}(\theta)$	

TABLE 1. Definitions of the W-algebraic categories \mathcal{O}

Note. The full subcategories on which the centre $Z(\mathfrak{g}, e)$ acts by a given generalised central character ξ are denoted $\hat{\mathcal{O}}(\xi, e, \mathfrak{p})$, and so on.

Remark 4.2. There are a number of equivalent ways of phrasing the above conditions, some of which are used in Loseu's original paper.

Condition **O1** is equivalent to:

- (O1') $U(\mathfrak{g}, e)_{>0}$ acts by locally nilpotent endomorphisms.

Further, conditions **O1** and **O2** together are equivalent to condition **O1** and:

- (O2') The $U(\mathfrak{g}, e)^0$ -module obtained after taking $U(\mathfrak{g}, e)_{>0}$ -invariants is of finite dimension, where $U(\mathfrak{g}, e)^0 := U(\mathfrak{g}, e)_{\geq 0} / (U(\mathfrak{g}, e)U(\mathfrak{g}, e)_{>0} \cap U(\mathfrak{g}, e)_{\geq 0})$.

In addition, there are a number of full subcategories of $U(\mathfrak{g})\text{-mod}$ of interest to us; these are variations on subcategories of modules known as *generalised Whittaker modules*. To define these categories, we need to fix a choice of maximal nilpotent subalgebra \mathfrak{n} along with a character $\chi: \mathfrak{n} \rightarrow \mathbb{C}$; we can then define the shifted Lie algebra $\mathfrak{n}_\chi := \{\xi - \chi(\xi) : \xi \in \mathfrak{n}\}$. We will again consider full subcategories whose objects satisfy various subsets of the following conditions.

- (Wh1) The shifted Lie algebra \mathfrak{n}_χ acts by locally nilpotent endomorphisms.
- (Wh2) The action of the centre $Z(\mathfrak{g})$ is locally finite.
- (Wh3) The action of \mathfrak{t} on the module is semisimple.
- (Wh4) The action of $Z(\mathfrak{g})$ on the module is semisimple.

Theorem 4.3 ([Los3, Theorem 4.1], [Los1, Theorem 1.2.2(iii)], [Web, Proposition 7]). *There are equivalences between each of columns of table 1 and the corresponding columns of table 2. These equivalences still hold if one restricts to a given generalised character of $Z(\mathfrak{g})$ and the corresponding character of $Z(\mathfrak{g}, e)$.*

Conditions	1	1,2	1,3	1,2,3	1,2,4
Notation	$\text{Wh}(U(\mathfrak{g}), \mathfrak{n}_\chi)$				
Webster		$\bigoplus_\xi \widehat{\mathcal{O}}(\xi, \chi)$		$\bigoplus_\xi \mathcal{O}(\xi, \chi)$	$\bigoplus_\xi \mathcal{O}'(\xi, \chi)$
Loseu	$\widetilde{\text{Wh}}(e, \theta)$	$\text{Wh}(e, \theta)$	$\widetilde{\text{Wh}}^t(e, \theta)$	$\text{Wh}^t(e, \theta)$	

TABLE 2. Definitions of the Whittaker categories

Note. Here ξ ranges over the set of generalised central characters of $U(\mathfrak{g})$.

We shall provide a version of this theorem for which the Whittaker categories lie not in $U(\mathfrak{g})\text{-mod}$, but rather in $U(\mathfrak{g}, e')\text{-mod}$ for another nilpotent $e' \leq e$.

4.2. Equivariant Slodowy slices. The remaining results of this paper follow the techniques and methodology presented in Loseu's papers [Los1, Los2, Los3], but translated to the context of this paper. We present them here for clarity and to highlight the changes necessary in our situation.

Given a nilpotent element e with a good grading Γ given by the semisimple element h' , construct a Γ -graded \mathfrak{sl}_2 -triple (e, h, f) . Based on this data, Loseu defines the *equivariant Slodowy slice*:

$$(9) \quad \widetilde{\mathcal{S}}_\chi := G \times \mathcal{S}_\chi \subseteq G \times \mathfrak{g}^* \simeq T^*G.$$

This is a symplectic subvariety of the cotangent bundle T^*G , and is stable under a number of group actions. The group G acts on itself both on the left and right by multiplication, which induces corresponding Hamiltonian actions on the cotangent bundle. Loseu defines the following group actions:

- G acts by $g \cdot (g_1, \alpha) = (gg_1, \alpha)$.
- \mathbb{C}^\times acts by $t \cdot (g_1, \alpha) = (g_1 \gamma(t)^{-1}, t^{-2} \gamma(t) \alpha)$.

Here, $\gamma: \mathbb{C}^\times \rightarrow G$ is the cocharacter determined by exponentiation of the semisimple element h' . Recall also that G acts on \mathfrak{g}^* by the coadjoint action $(g\alpha)(\xi) = \alpha(\text{Ad}_{g^{-1}} \xi)$.

Choosing a Premet subgroup M for e , we further define an action of M on T^*G :

- M acts by $m \cdot (g_1, \alpha) = (g_1 m^{-1}, m\alpha)$.

This has moment map $\mu: G \times \mathfrak{g}^* \rightarrow \mathfrak{m}^*$ given by $\mu(g, \alpha) = \alpha|_{\mathfrak{m}}$. It is therefore clear that $T^*G \llbracket_\chi M \simeq \widetilde{\mathcal{S}}_\chi$, where this is the usual symplectic Hamiltonian reduction.

Having translated the problem of Hamiltonian reduction of Slodowy slices as Poisson varieties to that of Hamiltonian reduction of equivariant Slodowy slices as symplectic varieties, we can now state the following theorem.

Theorem 4.4. *The homomorphism $U(\mathfrak{g}, e_1) \llbracket_\kappa \mathfrak{k} \rightarrow U(\mathfrak{g}) \llbracket_{\chi_2} \mathfrak{m}_2$ of theorem 3.9 is an isomorphism.*

Proof. After taking G -invariants of this symplectic reduction, we obtain the previous Poisson reduction of Slodowy slices. By corollary 2.7, it therefore suffices to prove that the quantum Hamiltonian reduction induces an isomorphism of the classical Hamiltonian symplectic reductions. However this follows by classical symplectic Hamiltonian reduction by stages for semidirect products, which can be found in, e.g. [MMO⁺, Theorem 4.2.2]. \square

Recall now the constructions of section 3. For any $e_1 \in \mathfrak{sl}_n$ and any nilpotent orbit \mathcal{O}_2 which covers the orbit of e_1 , we produce a Premet subalgebra \mathfrak{m}_1 , a nilpotent $e_2 \in \mathcal{O}_2$, a subalgebra \mathfrak{m}_2 , and a semisimple element h'_2 which gives a grading Γ which is good for e_2 . Choosing (e_2, h_2, f_2) to be a Γ -graded \mathfrak{sl}_2 -triple, we can define

the Slodowy slice \mathcal{S}_{χ_2} . The reduction by stages construction of section 3 therefore produces the following commutative diagram:

$$(10) \quad \begin{array}{ccc} T^*G & \xlongequal{\quad} & T^*G \\ \uparrow & & \uparrow \\ G \times (\chi_2 + \mathfrak{m}_2^\perp) & \xrightarrow{\iota} & G \times (\chi_1 + \mathfrak{m}_1^\perp) \\ \uparrow & & \uparrow \\ \mathcal{S}_{\chi_2} & \xrightarrow{\varphi} & \mathcal{S}_{\chi_1} \end{array}$$

Here, the vertical maps $G \times (\chi_i + \mathfrak{m}_i^\perp) \hookrightarrow T^*G$ and $G \times (\chi_i + \mathfrak{m}_i^\perp) \rightarrow \widetilde{\mathcal{S}}_{\chi_i}$ are the natural maps coming from Hamiltonian reduction, while the inclusions of \mathcal{S}_{χ_i} come from the natural presentation $\mathcal{S}_{\chi_i} = \chi_i + (\mathfrak{g}/[\mathfrak{g}, f_i])^* \subseteq \chi_i + \mathfrak{m}_i^\perp$. The map ι is the natural extension of the inclusion $\chi_2 + \mathfrak{m}_2^\perp \hookrightarrow \chi_1 + \mathfrak{m}_1^\perp$, and φ is defined as the obvious composition of maps.

In this context, we shall consider the following additional actions on T^*G , which preserve each of $\widetilde{\mathcal{S}}_{\chi_i}$:

- $Q := Z_G(e_1, h_1, f_1) \cap Z_G(e_2, h_2, f_2) \cap Z_G(h'_2)$ acts by $g_0 \cdot (g_1, \alpha) = (g_1 g_0^{-1}, g_0 \alpha)$.
- $\widetilde{G} := G \times \mathbb{C}^\times \times Q$, which acts component-wise.

Lemma 4.5. *The map φ is a \widetilde{G} -equivariant embedding of symplectic manifolds.*

Proof. That φ is $(G \times Q)$ -equivariant is manifest from the construction. To see that φ is injective, note that $\varphi(x) = \varphi(y)$ if and only if $\iota(x)$ and $\iota(y)$ lie in the same M_1 -orbit. But then x and y lie in the same M_2 -orbit, as $M_1 \subseteq M_2$, which would imply that $x = y$.

To see that φ is symplectic, note that the M_i -orbits form a nilfoliation of $\widetilde{\mathcal{S}}_{\chi_i}$ in $G \times (\chi_i + \mathfrak{m}_i^\perp)$. Hence, lifting along the M_2 -orbits can equally well be accomplished by restricting to lifting along M_1 -orbits, and so the symplectic forms will agree.

As in the classical case, the cocharacter γ associated to the semisimple element h'_2 gives an action of \mathbb{C}^\times on $\widetilde{\mathcal{S}}_{\chi_2}$. By lemma 3.12, the adjoint action $\text{Ad}_{\gamma(t)}$ stabilises \mathfrak{m}_1 , and hence this also gives a well-defined action on \mathcal{S}_{χ_1} . The map φ intertwines these two actions, and in both cases scales the symplectic form: $t \cdot \omega = t^2 \omega$. \square

Putting these facts together yields the following theorem.

Theorem 4.6. *For any pair of nilpotent elements $e_1 \leq e_2$ in \mathfrak{sl}_n in the dominance ordering, there is a $(G \times Q)$ -equivariant embedding of symplectic manifolds $\mathcal{S}_{\chi_2} \hookrightarrow \mathcal{S}_{\chi_1}$. Taking G -invariants yields an embedding of Poisson manifolds $\mathcal{S}_{\chi_2} \hookrightarrow \mathcal{S}_{\chi_1}$. Furthermore, there exist \mathbb{C}^\times -actions on both sides, intertwined by the embedding, which scale the symplectic forms (resp. Poisson bivectors) by a factor of t^2 . This \mathbb{C}^\times -action is a contracting action on \mathcal{S}_{χ_2} .*

4.3. The decomposition lemma. Consider the point $x = (1, \chi_2)$; the embedding of theorem 4.6 induces an inclusion of symplectic vector spaces $\varphi_*: T_x \mathcal{S}_{\chi_2} \hookrightarrow T_x \mathcal{S}_{\chi_1}$. This, in turn, induces an inclusion $(\mathfrak{g}/[\mathfrak{g}, f_2])^* \hookrightarrow (\mathfrak{g}/[\mathfrak{g}, f_1])^*$, and we let W denote its image. Finally, we define the subspace $V \subseteq \mathfrak{z}_{\mathfrak{g}}(e_1)$ as follows:

$$(11) \quad V := W^\perp \cap \mathfrak{z}_{\mathfrak{g}}(e_1) = \{\xi \in \mathfrak{z}_{\mathfrak{g}}(e_1) : \alpha(\xi) = 0 \text{ for all } \alpha \in W\}.$$

This vector space has a symplectic form expressed by $\omega(x, y) = \chi_2([x, y])$. Note that if $e_1 = 0$, then $V = [\mathfrak{g}, f_2]$.

The symplectic form on $T_x \widetilde{\mathcal{S}}_{\chi_1}$ is given by the expression

$$\omega(\xi + \alpha, \eta + \beta) = \chi_2([\xi, \eta]) - \langle \xi, \beta \rangle + \langle \eta, \alpha \rangle,$$

so the symplectic complement $(\varphi_* T_x \widetilde{\mathcal{S}}_{\chi_2})^{\perp_\omega}$ is $\{(\xi, \text{ad}_\xi^* \chi_2) : \xi \in V\}$. Projecting onto the first component identifies this with V , but we'll instead identify the it with V^* ; they are isomorphic as symplectic Q -modules. This gives a $(\mathbb{C}^\times \times Q)$ -equivariant symplectic isomorphism $\psi: T_x \mathcal{S}_{\chi_2} \oplus V^* \rightarrow T_x \mathcal{S}_{\chi_1}$, given by $\psi(v, w) = \varphi_*(v) + w$.

The standard considerations of Fedosov quantisation (cf. [Los1, Section 2.2]) yield $(G \times Q)$ -invariant, homogeneous, degree 2 star products on each of $\mathbb{C}[\widetilde{\mathcal{S}}_{\chi_i}][\hbar]$, the Moyal-Weyl star product on $\mathbb{C}[V^*][\hbar]$, and finally on the product $\mathbb{C}[\mathcal{S}_{\chi_2} \times V^*][\hbar]$. Since the star products are differential, they induce star products on the completions $\mathbb{C}[\widetilde{\mathcal{S}}_{\chi_1}]_{Gx}^\wedge[\hbar]$ and $\mathbb{C}[\mathcal{S}_{\chi_2} \times V^*]_{Gx}^\wedge[\hbar]$. Applying the argument of [Los1, Theorem 3.3.1] yields a \widetilde{G} -equivariant $\mathbb{C}[\hbar]$ -algebra isomorphism

$$(12) \quad \Phi_\hbar: \mathbb{C}[\widetilde{\mathcal{S}}_{\chi_1}]_{Gx}^\wedge[\hbar] \rightarrow \mathbb{C}[\mathcal{S}_{\chi_2} \times V^*]_{Gx}^\wedge[\hbar].$$

Taking G -invariants produces the following analogue of [Los3, Proposition 2.1].

Theorem 4.7. *There is a $(\mathbb{C}^\times \times Q)$ -equivariant $\mathbb{C}[\hbar]$ -algebra isomorphism*

$$\Phi_\hbar: \mathbb{C}[\mathcal{S}_{\chi_1}]_{\chi_2}^\wedge[\hbar] \rightarrow \mathbb{C}[\mathcal{S}_{\chi_2}]_{\chi_2}^\wedge[\hbar] \widehat{\otimes}_{\mathbb{C}[\hbar]} \mathbb{C}[V^*]_0^\wedge[\hbar]$$

satisfying:

- (1) $\Phi_\hbar(\sum_{i=0}^\infty f_i \hbar^{2i})$ contains only even powers of \hbar .
- (2) The cotangent map $d_0(\Phi_\hbar)^*: \mathfrak{z}_\mathfrak{g}(e_2) \otimes V \rightarrow \mathfrak{z}_\mathfrak{g}(e_1)$ co-incides with ψ .
- (3) For ι_1, ι_2 the respective embeddings of \mathfrak{q} into the domain and codomain of Φ_\hbar , then $\Phi_\hbar \circ \iota_1 = \iota_2$.

4.4. Loseu's machinery. In order to continue, we recall the machinery Loseu has developed for proving theorem 4.3. Let the following be given.

- $\mathfrak{v} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{v}(i)$ is a graded finite-dimensional vector space on which a torus T acting by preserving the grading.
- $A := \text{Sym}(\mathfrak{v})$, with the induced grading $A = \bigoplus_{i \in \mathbb{Z}} A_i$ and induced T -action.
- (\mathcal{A}, \circ) is an algebra with the same underlying vector space as A , where the algebra structure comes from a T -invariant deformation quantisation.
- ω_1 is a symplectic form on $\mathfrak{v}(1)$, where $\omega_1(u, v)$ is the constant term of the commutator in \mathcal{A} , and \mathfrak{v} is a lagrangian subspace of $\mathfrak{v}(1)$.
- $\mathfrak{m} := \mathfrak{v} \oplus \bigoplus_{i \leq 0} \mathfrak{v}(i)$.
- v_1, v_2, \dots, v_n is a homogeneous basis of \mathfrak{v} such that v_1, v_2, \dots, v_m form a basis of \mathfrak{m} . Further, let d_i be the degree of v_i and assume that d_1, d_2, \dots, d_m are increasing and that all v_i are T -semi-invariant.
- A^\heartsuit is the subalgebra of $\mathbb{C}[[\mathfrak{v}^*]]$ consisting of elements of the form $\sum_{i \leq c} f_i$ for some c , where f_i is a homogeneous power series of degree i .
- \mathcal{A}^\heartsuit is the algebra A^\heartsuit with multiplication as in \mathcal{A} . Any element of \mathcal{A}^\heartsuit can be written as an infinite linear combination of monomials $v_{i_1} \circ \dots \circ v_{i_\ell}$, where $i_1 \geq \dots \geq i_\ell$ and $\sum_{j=1}^\ell d_{i_j} \leq c$ for some c . Hence there is a filtration $F_c \mathcal{A}^\heartsuit$.
- θ is a co-character of T , and $\mathfrak{v}_{\geq 0}$ and $\mathfrak{v}_{> 0}$ are, respectively, the sums of the positive and strictly-positive $\text{ad } \theta$ -eigenspaces of \mathfrak{v} . We shall further require that $\mathfrak{v}_{> 0} \subseteq \mathfrak{m} \subseteq \mathfrak{v}_{\geq 0}$.
- $\mathcal{A}_{\geq 0}, \mathcal{A}_{> 0}, \mathcal{A}_{\geq 0}^\heartsuit, \mathcal{A}_{> 0}^\heartsuit$ are all defined analogously.
- $\mathcal{A}^\wedge := \lim \mathcal{A}/\mathcal{A}\mathfrak{m}^k$. Note that there is an injective algebra homomorphism $\mathcal{A}^\heartsuit \rightarrow \mathcal{A}^\wedge$.

Proposition 4.8. [Los3, Proposition 5.1] *Let (\mathcal{A}, \circ) and (\mathcal{A}', \circ') be two different algebras coming from A and \mathfrak{v} as above. Suppose there is a subspace $\mathfrak{v} \subseteq \mathfrak{v}(1)$ which is Lagrangian for both symplectic forms, and every element of A can be written as a finite sum of monomials in both \mathcal{A} and \mathcal{A}' . Then any homogeneous T -equivariant isomorphism $\Phi: \mathcal{A}^\heartsuit \rightarrow \mathcal{A}'^\heartsuit$*

satisfying $\Phi(v_i) - v_i \in F_{d_i-2}\mathcal{A} + (F_{d_i}\mathcal{A} \cap \mathfrak{v}^2\mathcal{A})$ extends uniquely to a topological algebra isomorphism $\Phi: \mathcal{A}^\wedge \rightarrow \mathcal{A}'^\wedge$ with $\Phi(\mathcal{A}^\wedge \mathfrak{m}) = \mathcal{A}'^\wedge \mathfrak{m}$.

Corollary 4.9. [Los3, Corollary 5.2] *The isomorphism $\Phi: \mathcal{A}^\wedge \rightarrow \mathcal{A}'^\wedge$ induces an equivalence of categories $\Phi_*: \text{Wh}(\mathcal{A}, \mathfrak{m}) \rightarrow \text{Wh}(\mathcal{A}', \mathfrak{m})$, where $\text{Wh}(\mathcal{A}, \mathfrak{m})$ is the category of \mathcal{A}^\wedge -modules which are annihilated by some $\mathcal{A}^\wedge \mathfrak{m}^k$. This equivalence preserves the subcategories on which \mathfrak{t} acts semisimply, and commutes with the functor of taking \mathfrak{m} -invariants; i.e. $\Phi_*(M^\mathfrak{m}) = \Phi_*(M)^\mathfrak{m}$.*

Note. $\text{Wh}(\mathcal{A}, \mathfrak{m})$ can naturally be viewed as the category of \mathcal{A} -modules on which \mathfrak{m} acts by locally nilpotent endomorphisms. This justifies the choice of notation in table 2.

With the constructions as before, we seek to make a set of choices which satisfy the hypotheses of proposition 4.8. First, we'll fix a maximal torus $T \subseteq Q$, and pick an arbitrary cocharacter θ , viewed as an element of \mathfrak{t} . Denote by \mathfrak{p} the parabolic subalgebra of \mathfrak{g} consisting of the positive eigenspaces of $\text{ad } \theta$. The zero eigenspace is a Levi subalgebra $\mathfrak{l} \subseteq \mathfrak{p}$, which contains each of e_i, h_i and f_i for $i = 1, 2$. The good grading Γ of \mathfrak{g} induces a good grading of \mathfrak{l} , and so one can pick a Premet subalgebra $\underline{\mathfrak{m}} \subseteq \mathfrak{l}$ as usual, with corresponding shift $\underline{\mathfrak{m}}_\chi$. Let $\tilde{\mathfrak{m}} := \underline{\mathfrak{m}} \oplus \mathfrak{g}_{>0}$, where $\mathfrak{g}_{>0}$ consists of the strictly positive eigenspaces of the action of $\text{ad } \theta$; the corresponding shift is $\tilde{\mathfrak{m}}_\chi := \underline{\mathfrak{m}}_\chi \oplus \mathfrak{g}_{>0}$.

- (1) Define $\mathfrak{v} := \{\xi - \chi_2(\xi) : \xi \in \mathfrak{z}(e_1)\}$. Note that $\mathfrak{v} \simeq \mathfrak{z}_\mathfrak{g}(e_2) \oplus V$, as shown in section 4.3.
- (2) Choosing $m > 2 + 2d$, where d is the maximum eigenvalue of $\text{ad } h'_2$ on \mathfrak{g} , define the grading on \mathfrak{v} to be given by

$$\mathfrak{v}(i) = \{\xi \in \mathfrak{v} : (\text{ad } h'_2 - m \text{ad } \theta)\xi = (i - 2)\xi\}.$$

- (3) Set \mathfrak{m} to be $\tilde{\mathfrak{m}}_\chi \cap \mathfrak{v}$, which satisfies $\mathfrak{v}_{>0} \subseteq \mathfrak{m} \subseteq \mathfrak{v}_{\geq 0}$ by our choice of m .
- (4) Define $\mathcal{A} := U(\mathfrak{g}, e_1)$ and $\mathcal{A}' = \mathbf{A}_V \otimes U(\mathfrak{g}, e_2)$.

These choices satisfy the conditions beginning this section. It remains to prove the following lemma.

Lemma 4.10. *There is an isomorphism $\Phi: U(\mathfrak{g}, e_1)^\heartsuit \rightarrow (\mathbf{A}_V \otimes U(\mathfrak{g}, e_2))^\heartsuit$ which satisfies the hypotheses of proposition 4.8.*

Proof. It follows from the considerations of section 4.3 that \mathcal{A} and \mathcal{A}' are both deformation quantisations of $A = \text{Sym}(\mathfrak{v})$. Further, since \mathcal{A}^\heartsuit and \mathcal{A}'^\heartsuit can be identified with the respective quotients by $\hbar - 1$ of the \mathbb{C}^\times -finite parts of

$$\mathbb{C}[\mathcal{S}_{\chi_1}]_{\chi_2}^\wedge[[\hbar]] \quad \text{and} \quad \mathbb{C}[\mathcal{S}_{\chi_2}]_{\chi_2}^\wedge[[\hbar]] \hat{\otimes}_{\mathbb{C}[[\hbar]]} \mathbb{C}[V^*]_0^\wedge[[\hbar]],$$

the isomorphism Φ_\hbar of theorem 4.7 provides the necessary map Φ . This satisfies the hypotheses of proposition 4.8 by the same considerations as before (cf. [Los1, Corollary 3.3.2]). \square

4.5. Category equivalences. Finally, we shall use this machinery to develop the category equivalences we need. First, there is an equivalence

$$\mathcal{K}': \text{Wh}(\mathcal{A}', \mathfrak{m}) \rightarrow \tilde{\mathcal{O}}(e_2, \mathfrak{p}), \quad \mathcal{K}'(M) := (M)^{\tilde{\mathfrak{m}} \cap V}.$$

To see that the image lies in $\tilde{\mathcal{O}}(e_2, \mathfrak{p})$, it suffices to note that $U(\mathfrak{g}, e_2)_{>0}$ is generated by the strictly positive eigenspaces of $\text{ad } \theta$, all of which lie in \mathfrak{m} by construction. That this functor is an equivalence follows from results on representations of Heisenberg algebras, and the fact that $\tilde{\mathfrak{m}} \cap V$ is a Lagrangian subspace of V .

Combining this with the equivalence of corollary 4.9 yields the following theorem, which is the main result of this section and generalises theorem 4.3.

Theorem 4.11. *There exists an equivalence of categories*

$$\mathcal{K}: \text{Wh}(U(\mathfrak{g}, e_1), \mathfrak{m}) \rightarrow \tilde{\mathcal{O}}(e_2, \mathfrak{p}).$$

Furthermore, \mathcal{K} induces the following embeddings of categories \mathcal{O} , along with their block decompositions:

$$\begin{aligned} \tilde{\mathcal{O}}(e_2, \mathfrak{p}) &\hookrightarrow \tilde{\mathcal{O}}(e_1, \mathfrak{p}) & \hat{\mathcal{O}}(e_2, \mathfrak{p}) &\hookrightarrow \hat{\mathcal{O}}(e_1, \mathfrak{p}) \\ \mathcal{O}(e_2, \mathfrak{p}) &\hookrightarrow \mathcal{O}(e_1, \mathfrak{p}) & \mathcal{O}'(e_2, \mathfrak{p}) &\hookrightarrow \mathcal{O}'(e_1, \mathfrak{p}). \end{aligned}$$

Proof. The equivalence \mathcal{K} is defined to be $\mathcal{K}' \circ \Phi_*$. Let \mathfrak{m}_1 and \mathfrak{m}_2 be as in section 3.2, $V_1 := [\mathfrak{g}, f_1]$, and V and \mathfrak{m} be as above. These functors can then be arranged into a commutative diagram.

$$(13) \quad \begin{array}{ccccc} \text{Wh}(U(\mathfrak{g}), (\tilde{\mathfrak{m}}_1)_{\chi_1}) & \xrightarrow{\sim} & \text{Wh}(\mathbf{A}_{V_1 \otimes U}(\mathfrak{g}, e_1), (\tilde{\mathfrak{m}}_1)_{\chi_1}) & \xrightarrow{\sim} & \tilde{\mathcal{O}}(e_1, \mathfrak{p}) \\ & \nearrow & & & \\ \text{Wh}(\mathbf{A}_{V_1 \otimes U}(\mathfrak{g}, e_1), (\tilde{\mathfrak{m}}_2)_{\chi_2}) & \xrightarrow{\sim} & \text{Wh}(\mathbf{A}_{V_1 \oplus V \otimes U}(\mathfrak{g}, e_2), (\tilde{\mathfrak{m}}_2)_{\chi_2}) & \xrightarrow{\sim} & \tilde{\mathcal{O}}(e_2, \mathfrak{p}) \\ \sim \downarrow (\cdot)^{\tilde{\mathfrak{m}}_2 \cap V_1} & & \sim \downarrow (\cdot)^{\tilde{\mathfrak{m}}_2 \cap V_1} & & \parallel \\ \text{Wh}(U(\mathfrak{g}, e_1), \mathfrak{m}) & \xrightarrow[\Phi_*]{\sim} & \text{Wh}(\mathbf{A}_{V \otimes U}(\mathfrak{g}, e_2), \mathfrak{m}) & \xrightarrow[\mathcal{K}']{\sim} & \tilde{\mathcal{O}}(e_2, \mathfrak{p}) \end{array}$$

Here, the equivalences between the first two columns of each row are the functors Φ_* of corollary 4.9 in the appropriate settings. The functors between the second and third columns are the appropriate analogues of the functor \mathcal{K}' , taking invariants with respect to $\tilde{\mathfrak{m}}_1 \cap V_1$, $\tilde{\mathfrak{m}}_2 \cap (V_1 \oplus V)$ and $\tilde{\mathfrak{m}} \cap V$, respectively. Since these are all respective Lagrangian subspaces of V_1 , $V_1 \oplus V$ and V , it follows that they are equivalences.

From the diagram, it can be seen that there is an embedding of categories \mathcal{O} ,

$$\tilde{\mathcal{O}}(e_2, \mathfrak{p}) \hookrightarrow \tilde{\mathcal{O}}(e_1, \mathfrak{p}).$$

Since the functor \mathcal{K} intertwines the actions of \mathfrak{t} and $Z(\mathfrak{g}, e)$, it induces an embedding of each of the above subcategories, and also their block decompositions with respect to generalised central characters. \square

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MATHEMATICAL SCIENCES INSTITUTE, AUSTRALIAN NATIONAL UNIVERSITY, ACTON ACT 2061, AUSTRALIA

E-mail address: smorgan@math.utoronto.ca

URL: www.math.utoronto.ca/smorgan